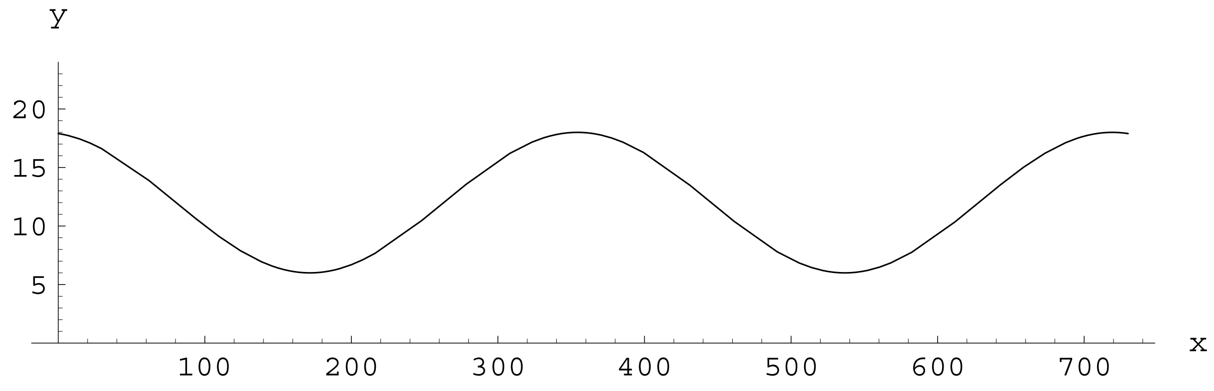
Periodic Functions

[§1.2 of the Notes, Chapter 1 of Stewart is also useful]

There are many examples in nature of events repeating themselves over and over again. Nature is periodic!

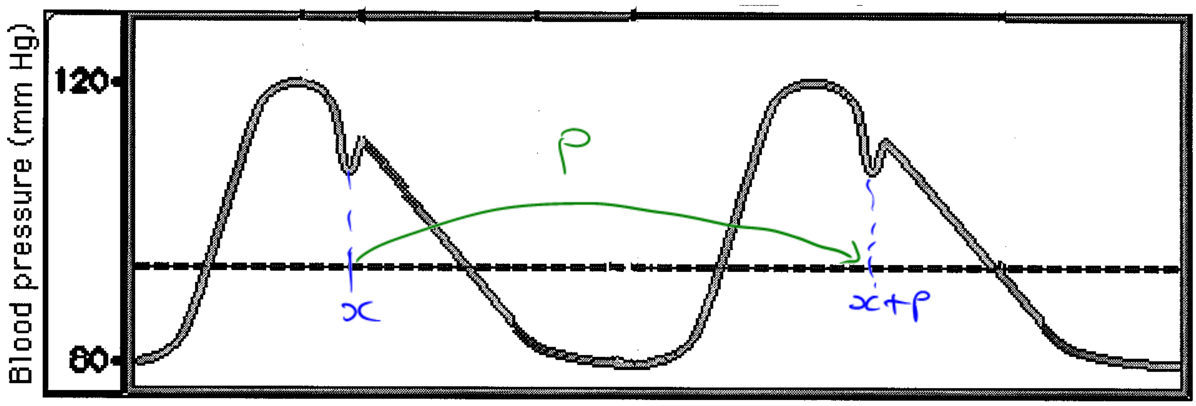
Example 1

If you plot the length of day against time the graph repeats itself at yearly intervals.



Example 2

If you plot blood pressure against time for a healthy person at rest the graph repeats itself.



A function, f, is periodic if for some number p (called the *period*),

$$f(x+\underline{p}) = f(x).$$

From this we can also see that

$$
\begin{aligned}
f(x+2p) &= f((x+p)+p) \\
&= f(x+p) \\
&= f(x).
\end{aligned}
$$

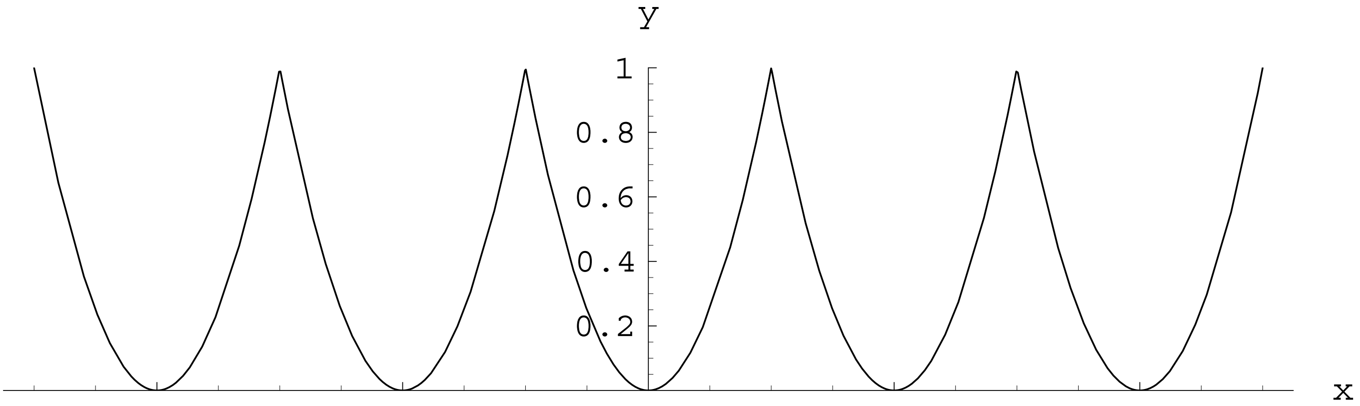
Furthermore, $f(x+3p) = f(x)$ and thus in general we have

Definition of a periodic function:

$$\boxed{f(x+kp) = f(x)}$$

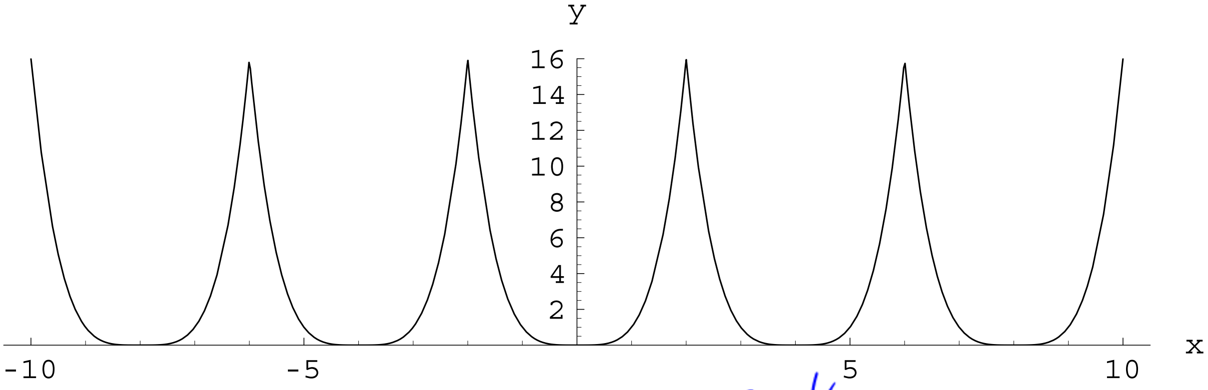
for all integers k.

This means that any piece of the graph of length p gives you the graph of the function by repetition.

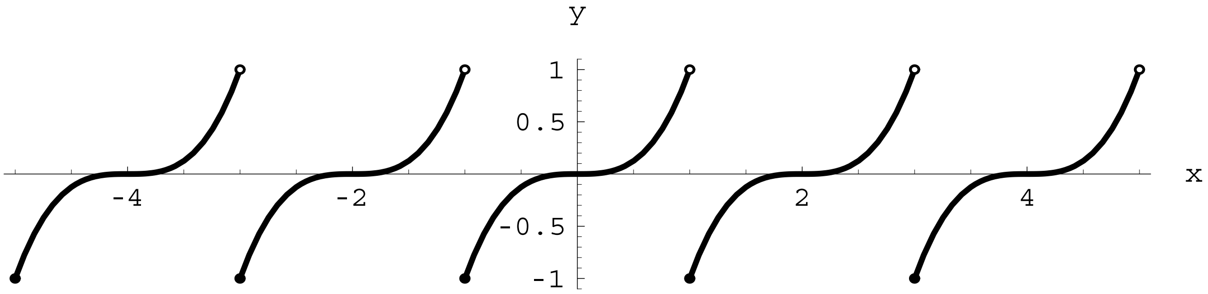


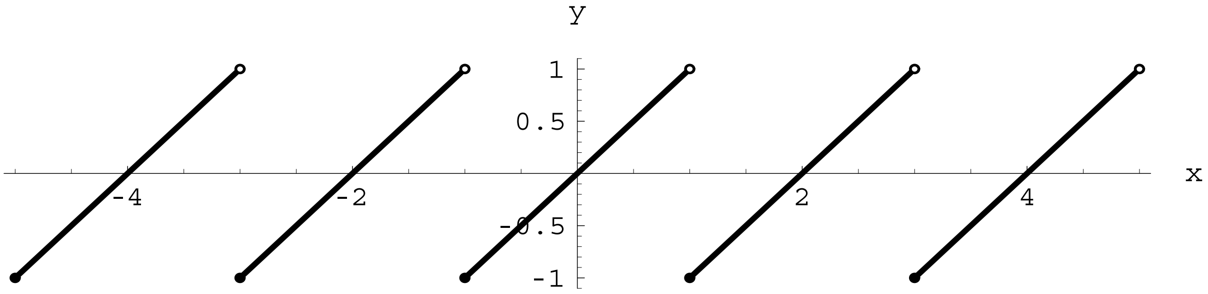
$p=2$

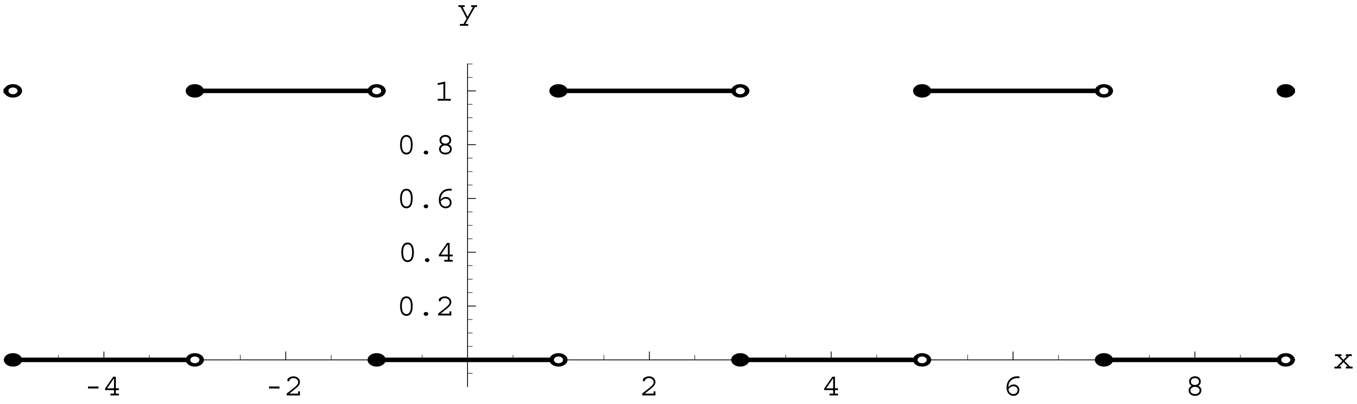
More examples



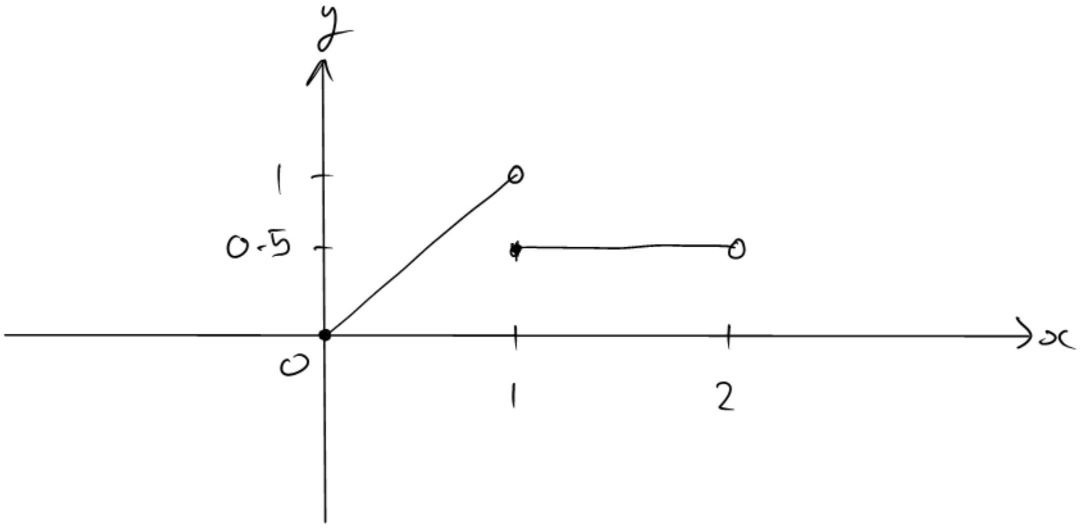
$p=4$


y
1
0.5
-0.5
-1
x
-4
-2
2
4


y
1
0.5
-0.5
-1
x
-4
-2
2
4


y
1
0.8
0.6
0.4
0.2
x
-4
-2
2
4
6
8

Eg: sketch $y = \begin{cases} x & 0 \leq x < 1 \\ 0.5 & 1 \leq x < 2 \end{cases}$

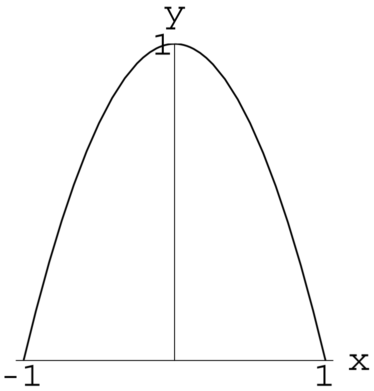


Typically, the recipe for a periodic function looks like this[1]:

$$f(x) = \begin{cases} 1 - x^2 \quad \text{p} & -1 < x \leq 1 \\ f(x + \textcircled{2}) & \text{for all } x. \end{cases}$$

tells you periodicity

To sketch the graph take the bit explicitly given by $1 - x^2$ where $-1 < x \leq 1$. Sketch this,



and copy it along by ± 2, ± 4,. . . etc.

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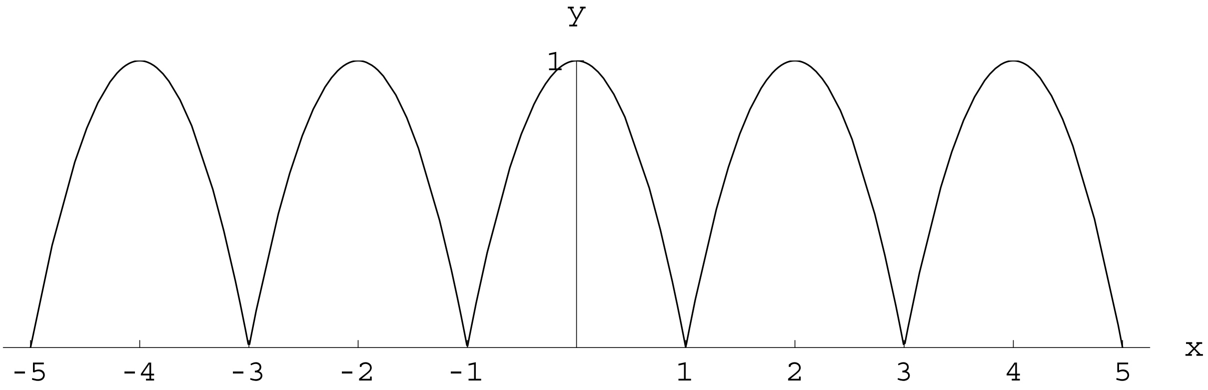
[1] You may want to review "Piecewise Defined Functions" in §1.1 p17 of Stewart.



To find values of the function

$$f(x) = \begin{cases} 1 - x^2 & -1 < x \leq 1 \\ f(x+2) & \text{all } x \end{cases}$$

you need to find an n such that

$$-1 < x + 2n \leq 1$$

and use the explicit part.

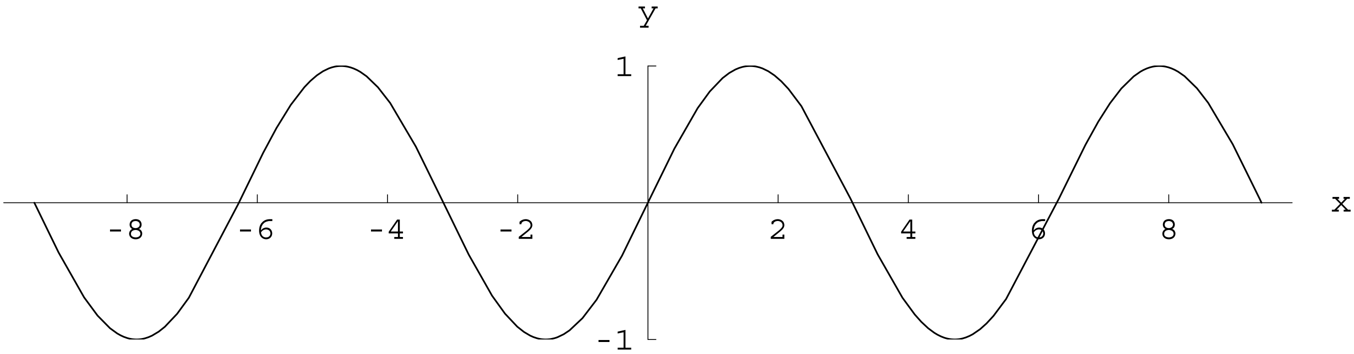
$$\begin{aligned}
f(25) &= f(\overset{x}{\boxed{1}} + \cancel{\overset{p}{2} \times \overset{n}{12}}) \\
&= f(1) = 0 \\
f(-3) &= f(1 + 2 \times (-2)) \\
&= f(1) = 0 \\
f(2.35) &= f(0.35 + 2 \times 1) \\
&= f(0.35) = 0.8775.
\end{aligned}$$

In other words you “*find the closest integer multiple of* p *to* x *and add or subtract to make the number smaller.*”

Trigonometric Functions

[*Stewart §1.2 pp32-33, §1.3 pp37-40, also revise Appendix D*]

We are now going to study functions corresponding to “sine-like” (or sinusoidal) curves[2]: curves obtained by moving the curve $y = \sin x$



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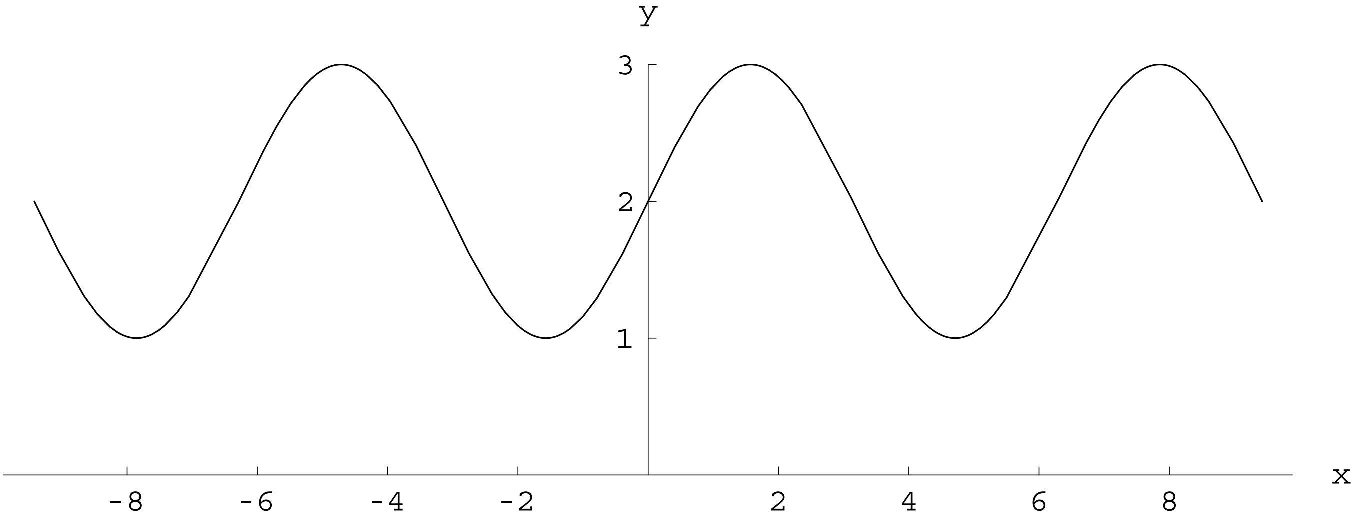
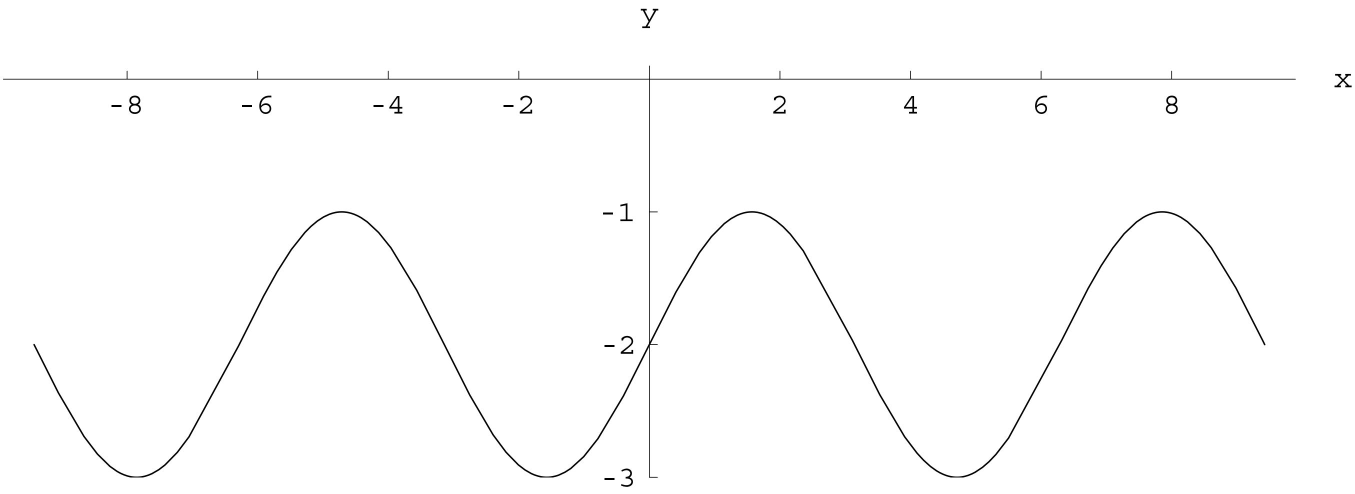
[2] You would have studied $\sin, \cos, \tan$ and their graphs in high school.

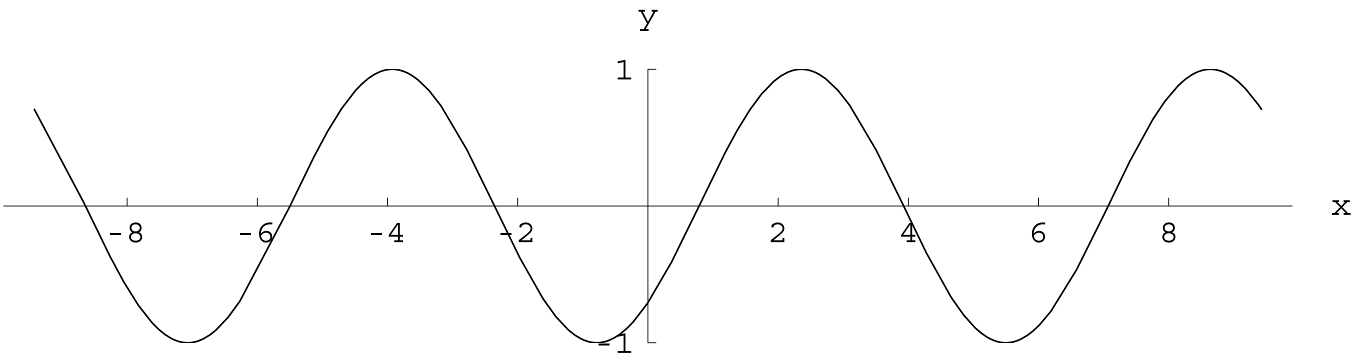
up



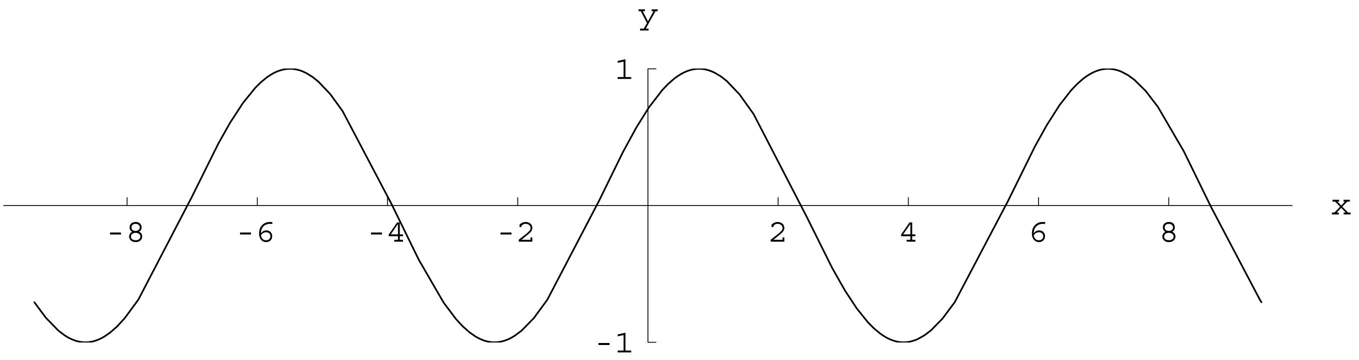
and down



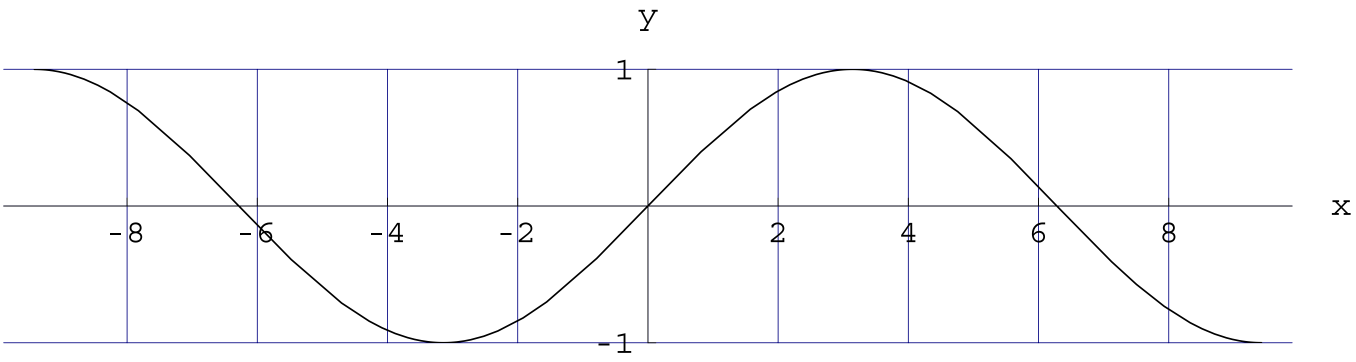
right to left



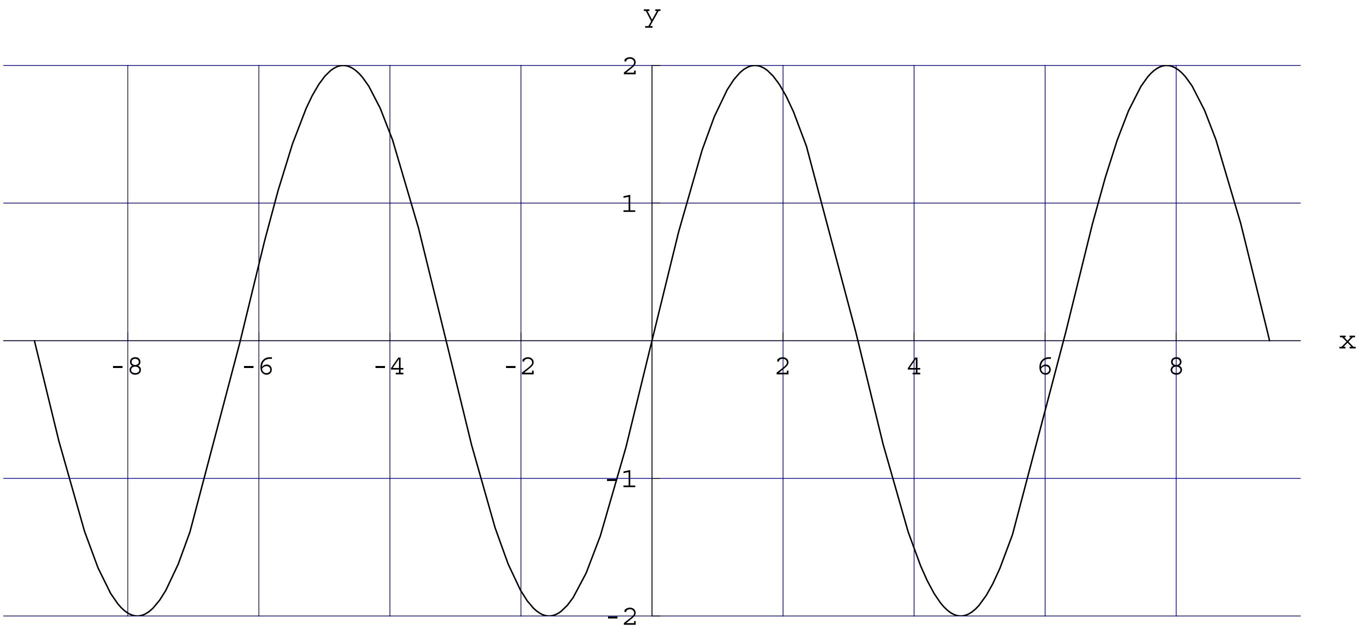
and left to right



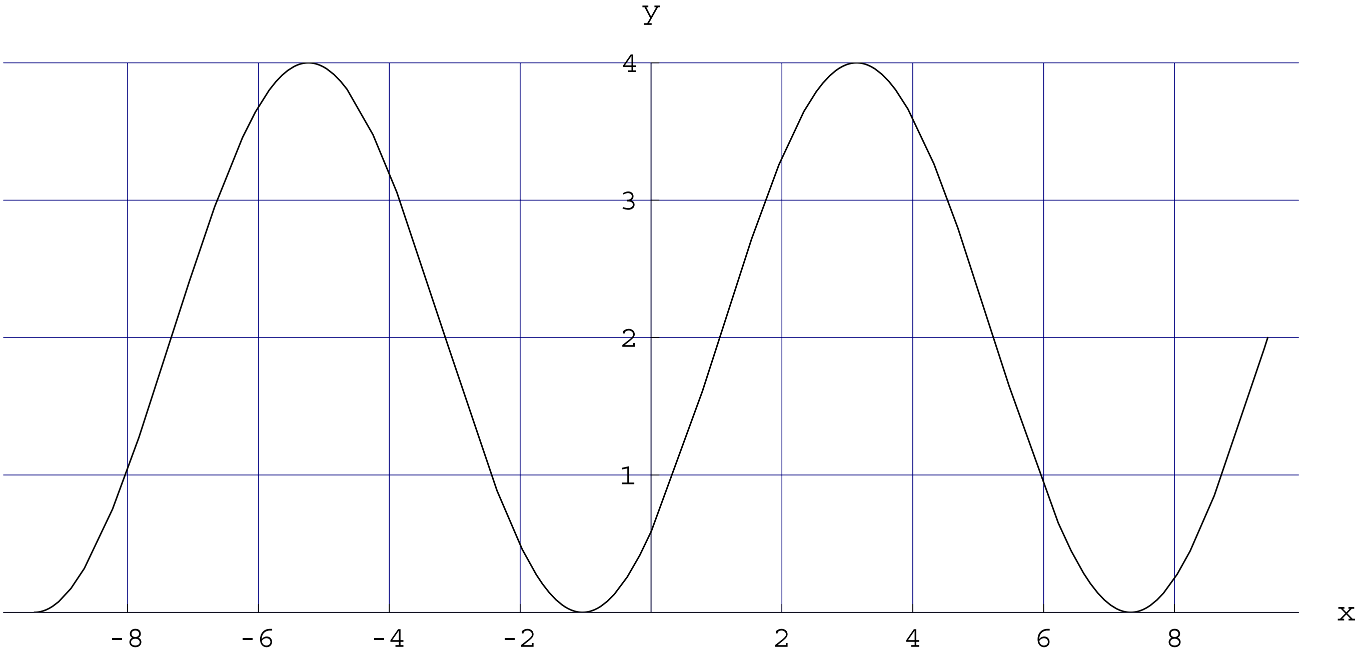
or stretching them horizontally



or vertically



or a combination of all the above.



Radians:

$2\pi = 360°$ $\frac{\pi}{4} = 45°$

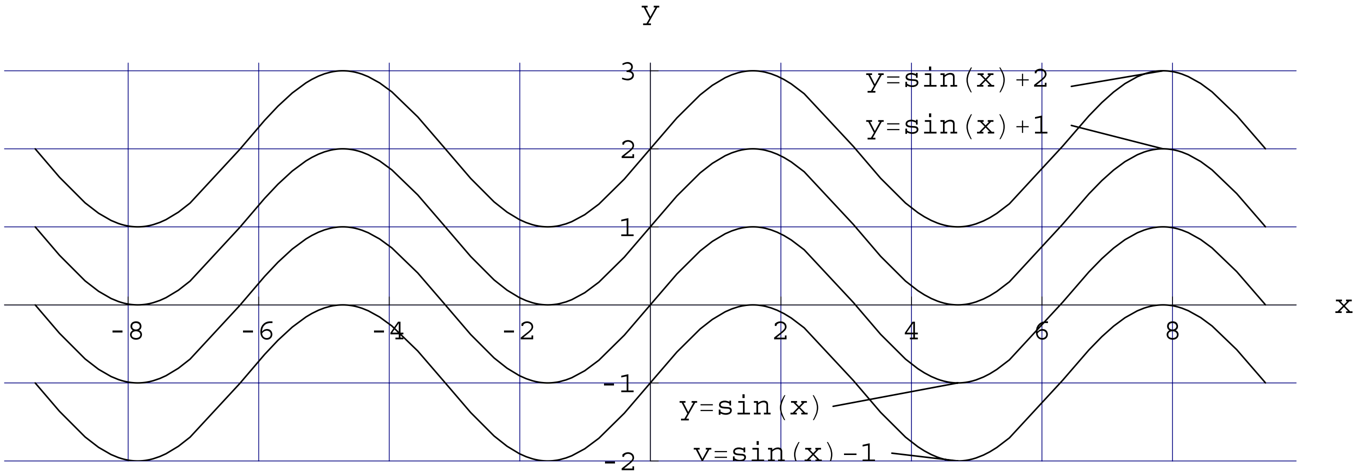
$\pi = 180°$

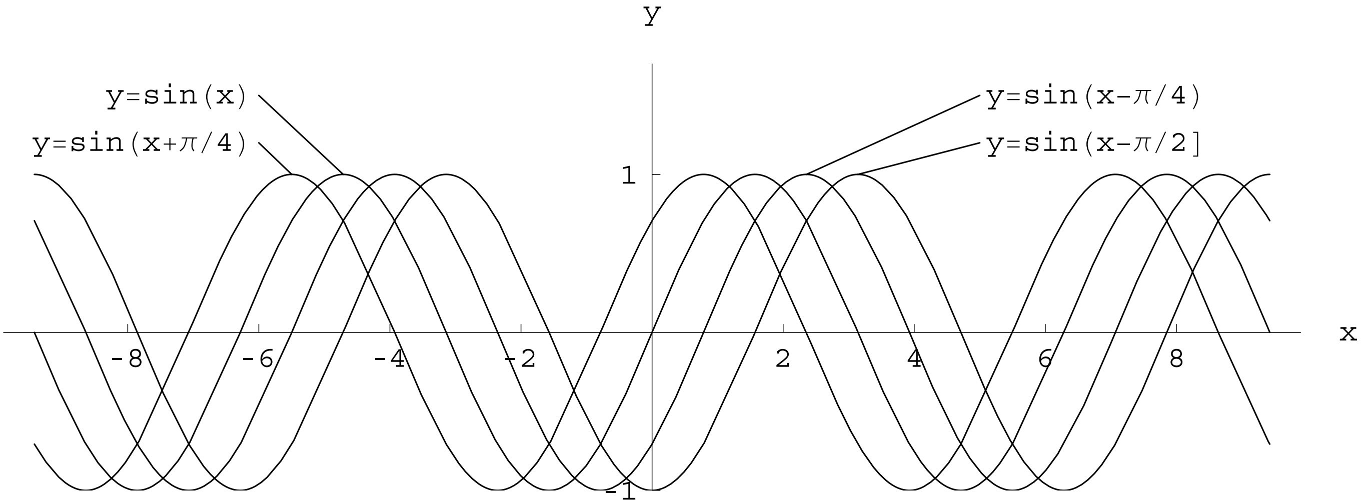
$\frac{\pi}{2} = 90°$ $\frac{\pi}{3} = 60°$

$\frac{\pi}{6} = 30°$

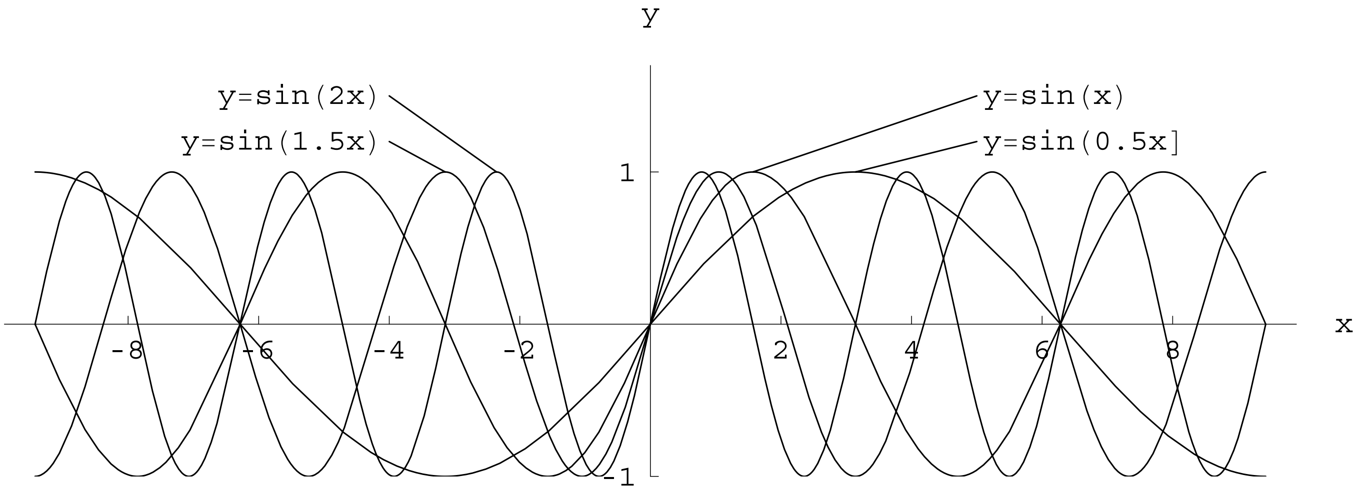
Here are some examples:

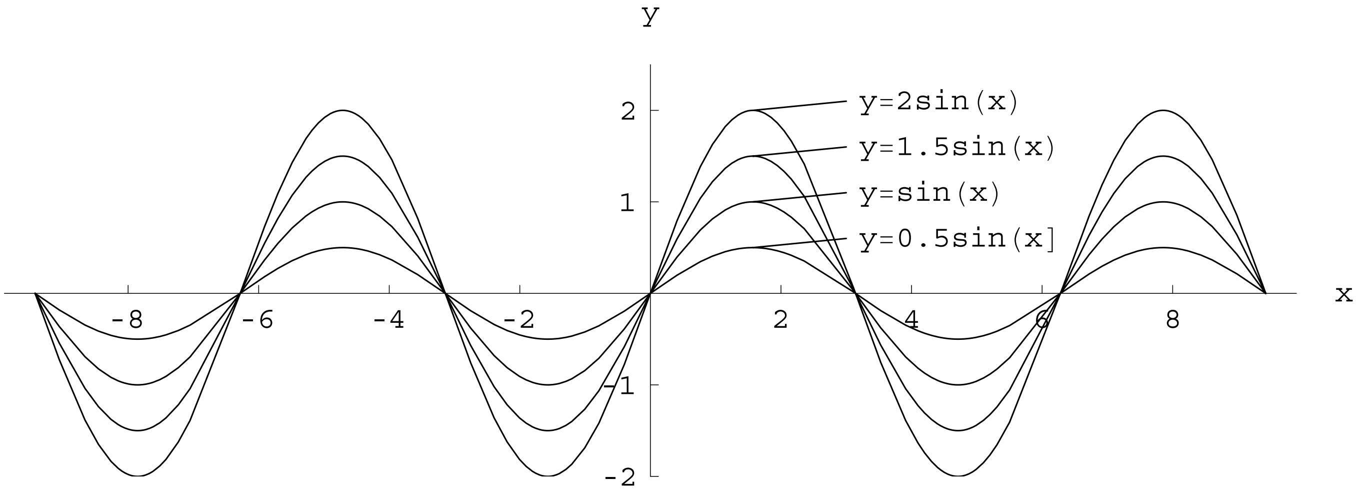
First of translating:





Then of stretching and squashing:

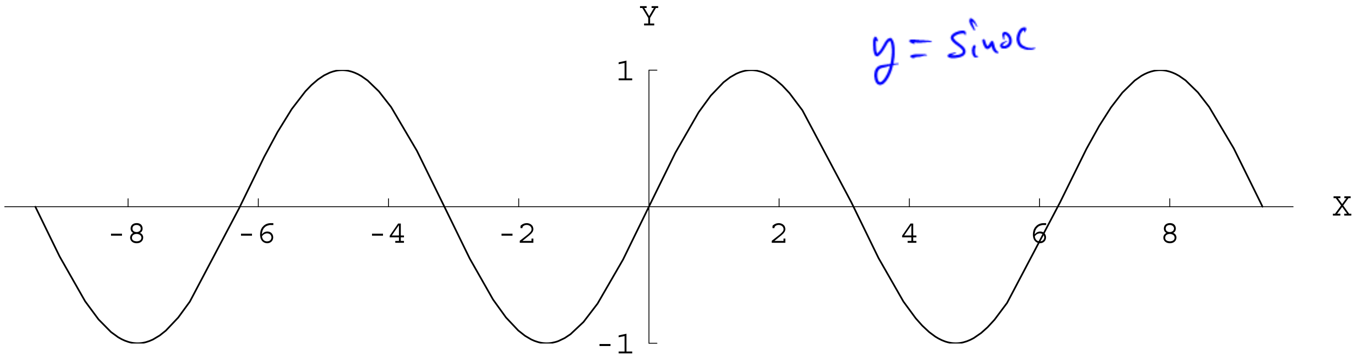




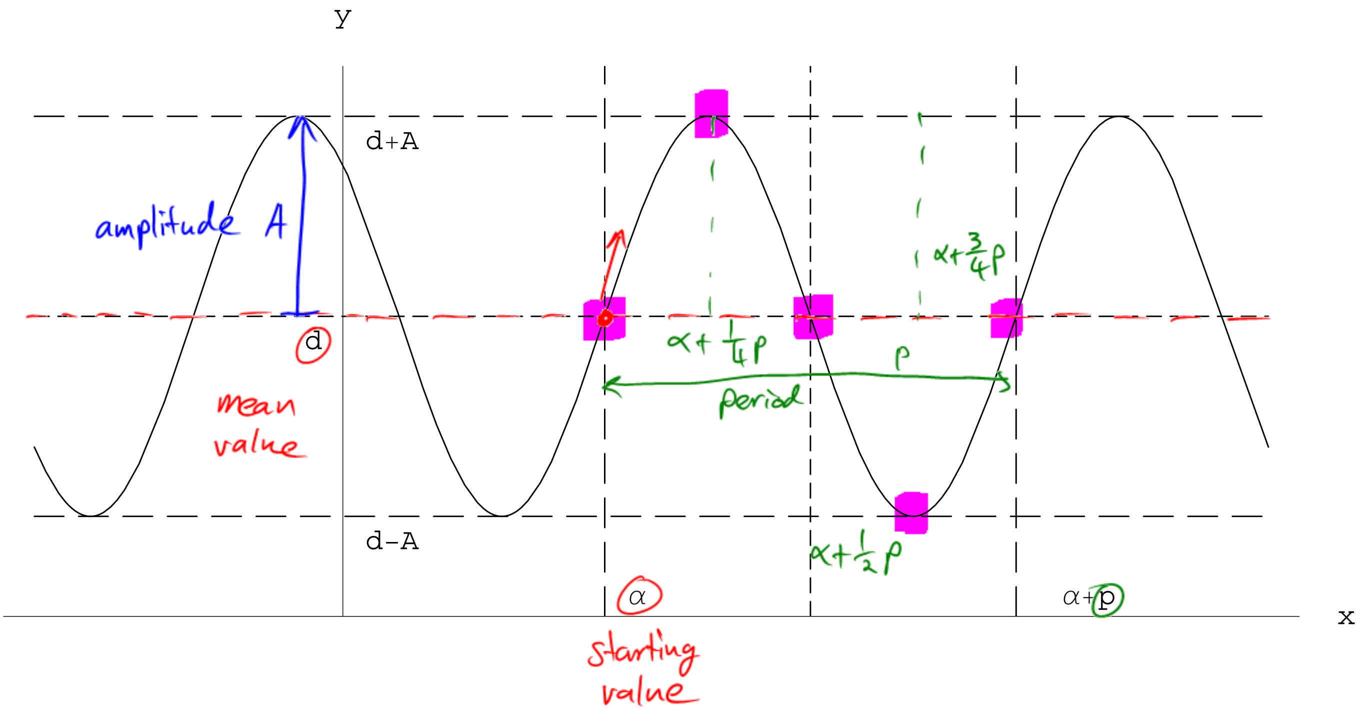
Here's how it goes in general.

Take a sine curve $y = \sin x$

- stretch/squash by a factor of A vertically
- stretch/squash by a factor of b horizontally
- move d up/down
- move α to the left/right.



Goes to



$$y = d + A\sin[b(x-\alpha)]$$

Derivation:

Start with $Y = \sin X$ and put:

$$\begin{cases} x = \dfrac{1}{b}X + \alpha \\ y = AY + d. \end{cases}$$

Then we have:

$$\begin{cases} X = b(x - \alpha) \\ Y = \dfrac{1}{A}(y - d). \end{cases}$$

Then the original sine curve

$$Y = \sin X$$

gives us the following relation between x and y

$$\frac{1}{A}(y - d) = \sin\left(b(x - \alpha)\right).$$

This, in turn gives

$$\boxed{y = d + A\sin\left[b(x - \alpha)\right],}$$

the equation of the stretched and shifted curve.

A, b, d and α are the "*vital statistics*" of the sinusoidal curve which results.

- A is called the amplitude
- $\boxed{p = \frac{2\pi}{b}}$ is the period
- d is the "mean value"
- α is the "starting point"

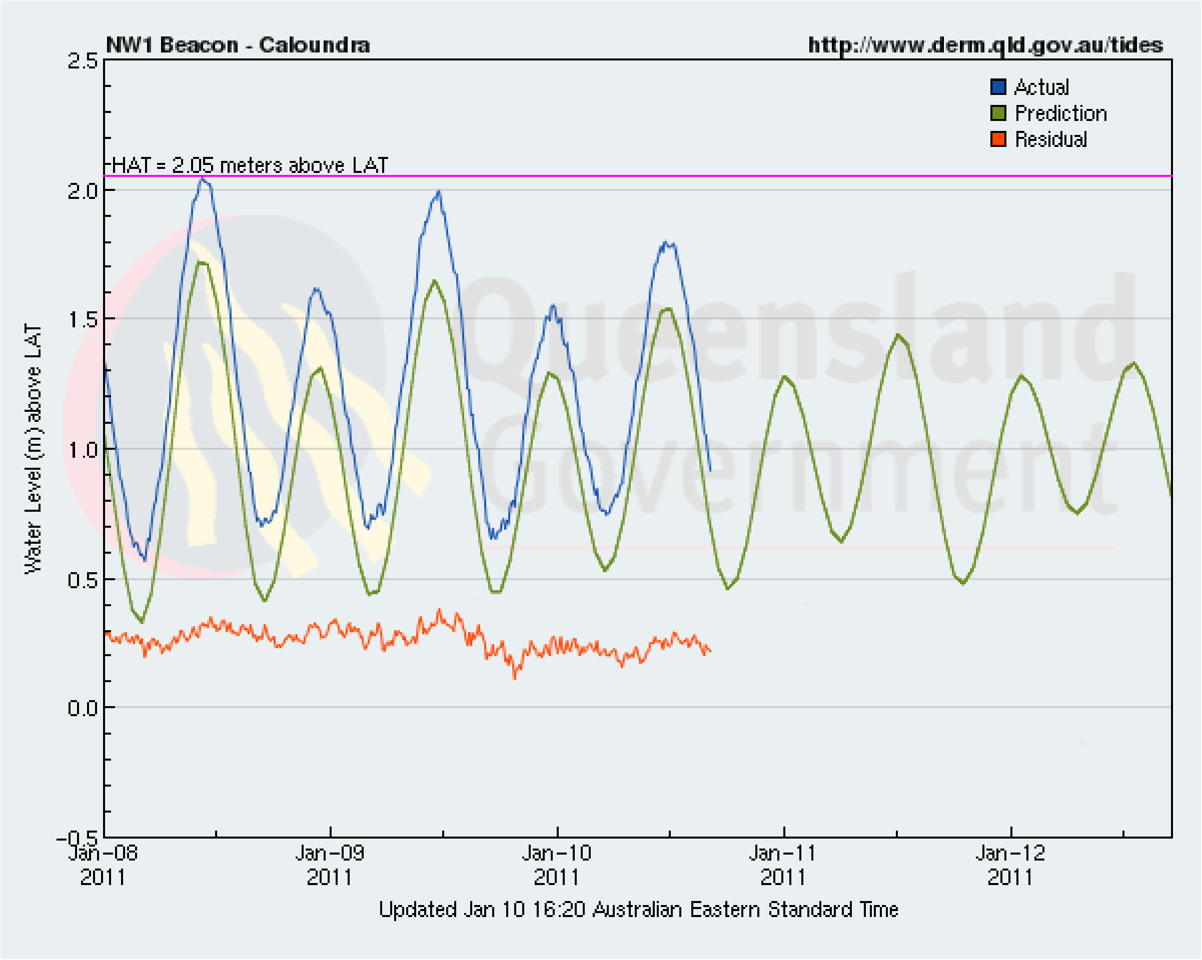
p is the period because

$$\begin{aligned} & d + A\sin\left[b\{(x+p) - \alpha\}\right] \\ & \quad = d + A\sin\left[b\left(x + \frac{2\pi}{b}\right) - b\alpha\right] \\ & \quad = d + A\sin\left[b(x-\alpha) + 2\pi\right] \\ & \quad = d + A\sin\left[b(x-\alpha)\right] \end{aligned}$$

Some additional properties:

- The $maximum\ value$ and $minimum\ value$ of a sinusoid are given by $d + A$ and $d - A$ respectively
- You can interpret the period as being the distance between two **adjacent** starting points or two maxima or two minima, but in any case it is the distance before the graph repeats itself
- The amplitude is the distance (hence is positive) between the maximum (or minimum) and the mean value

- The mean value d is also like a middle axis of symmetry about which the sinusoid oscillates about. Sometimes it's also called an "equilibrium". It is also the average of the max and min values.

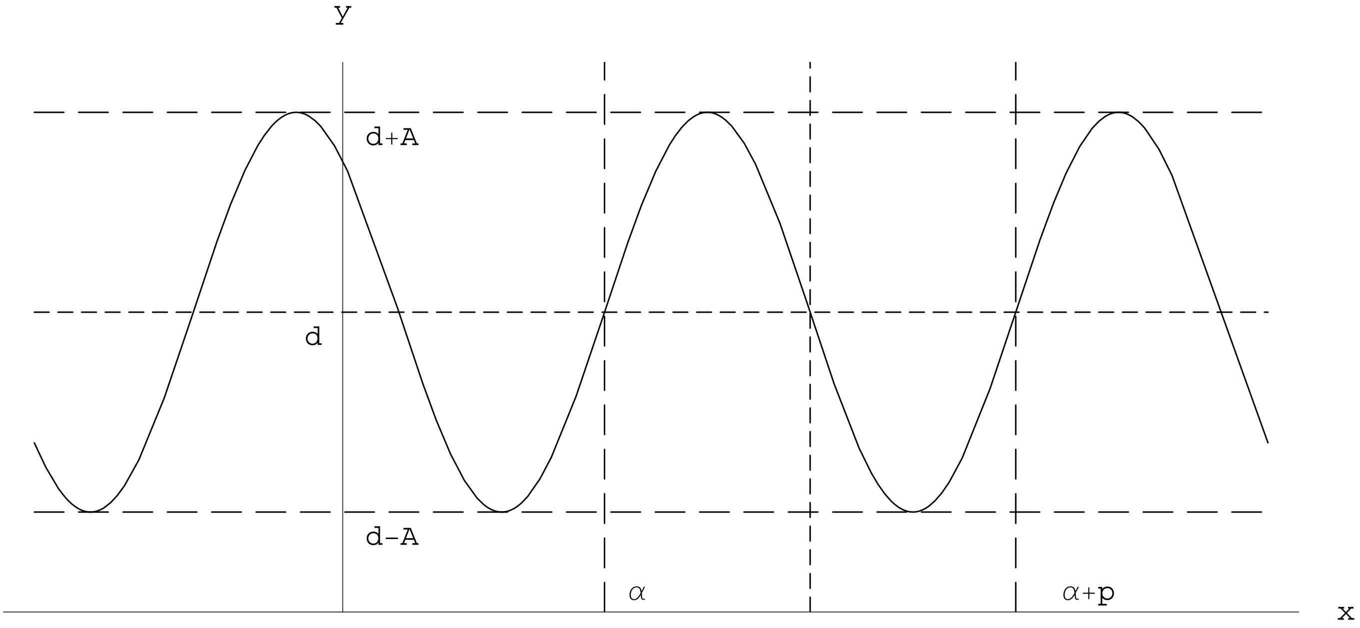


Example: Water Tides

Imagine you are looking at the water level on the side of a wharf. The water level would be approximately sinusoidal. The highest (lowest) water line is the maximum (minimum). In the middle would be the average water height. Half of the distance from the highest to lowest water lines would be the amplitude. Suppose it is high tide now. Then how long it takes to get back to high tide would be the period. The time when the tide is at the middle value and is *rising* would be the “starting value”.

Now we can sketch the graphs of sinusoidal functions by getting the vital statistics.

If we have the graph of a sinusoidal curve, we can estimate the vital statistics and write down the recipe for the corresponding function.



Example

Sketch[3] the graph of

$$\begin{aligned} y &= 3 + 2\sin(3x - 4) \\ &= 3 + 2\sin\left(3\left(x - \tfrac{4}{3}\right)\right) \end{aligned}$$

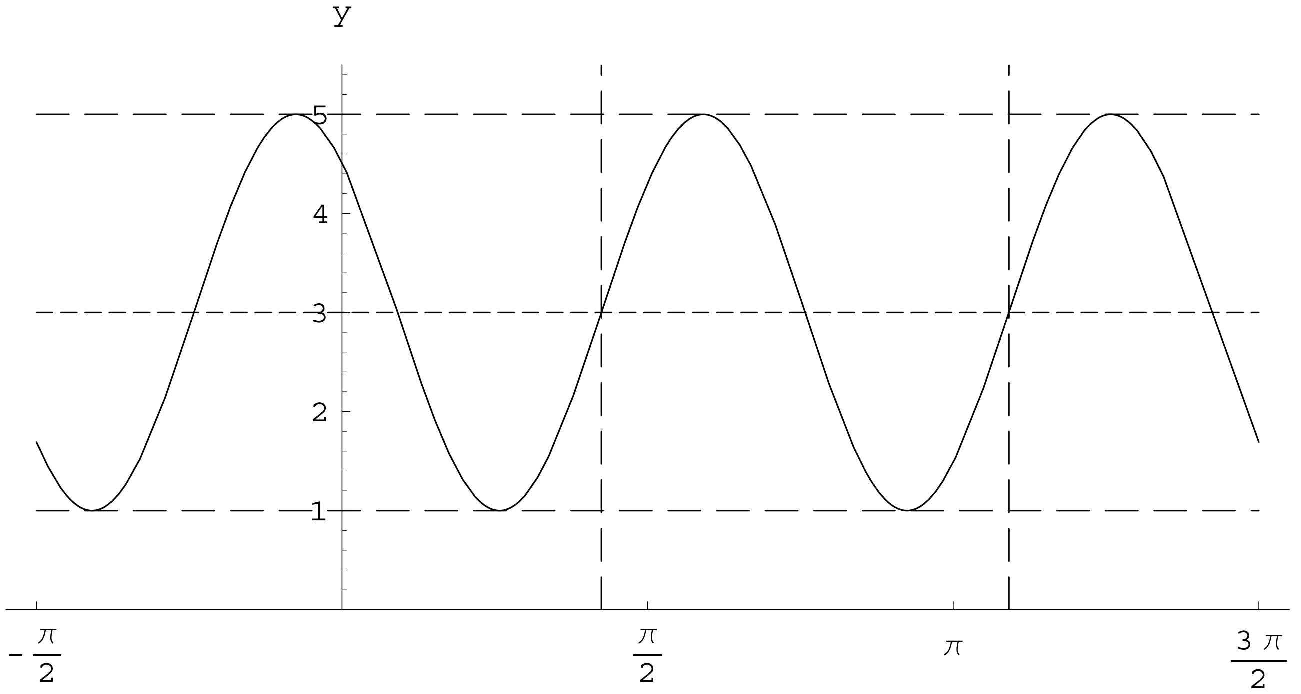
We have

amplitude 2
period $\frac{2\pi}{3}$
mean value 3
and start $\frac{4}{3}$.

---

[3] Notice here that we have factored out the 3 to get the equation in "standard form".

We mark the x-axis in multiples of π radians[4]. We have:



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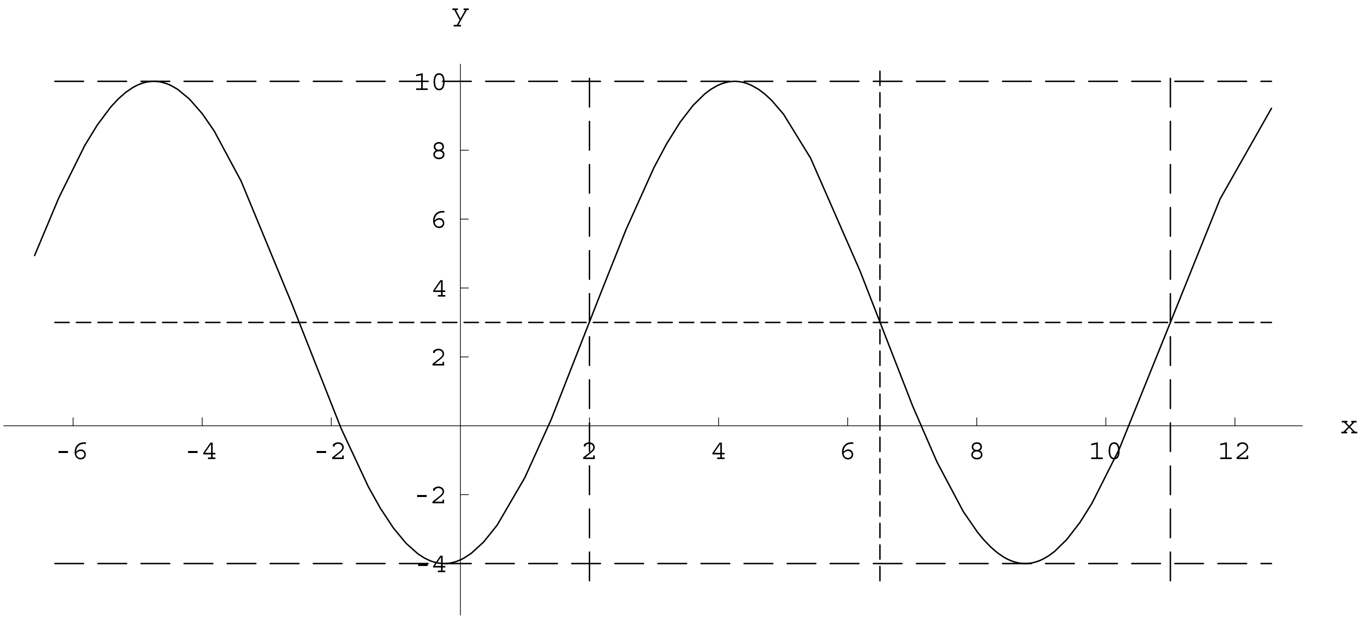
[4] You need to be familiar with the radian measure of angles as was taught at school as we will be using radians hereon in throughout the course. Quickly revise Appendix 3 of the *Notes*, and on ppA24-A25 of *Stewart*

Example

Given the following graph of a sinusoidal function,



write down an equation for the function.

We see the vital statistics

amplitude 7
period 9
mean value 3
and start 2

and write down an equation.

$$y = 3 + 7\sin\left[\tfrac{2\pi}{9}(x-2)\right].$$

Example

Sketch the graph of

$$\begin{aligned} y &= -1 + 3\sin(2x-4) \\ &= -1 + 3\sin\left[2\left(x-2\right)\right] \end{aligned}$$

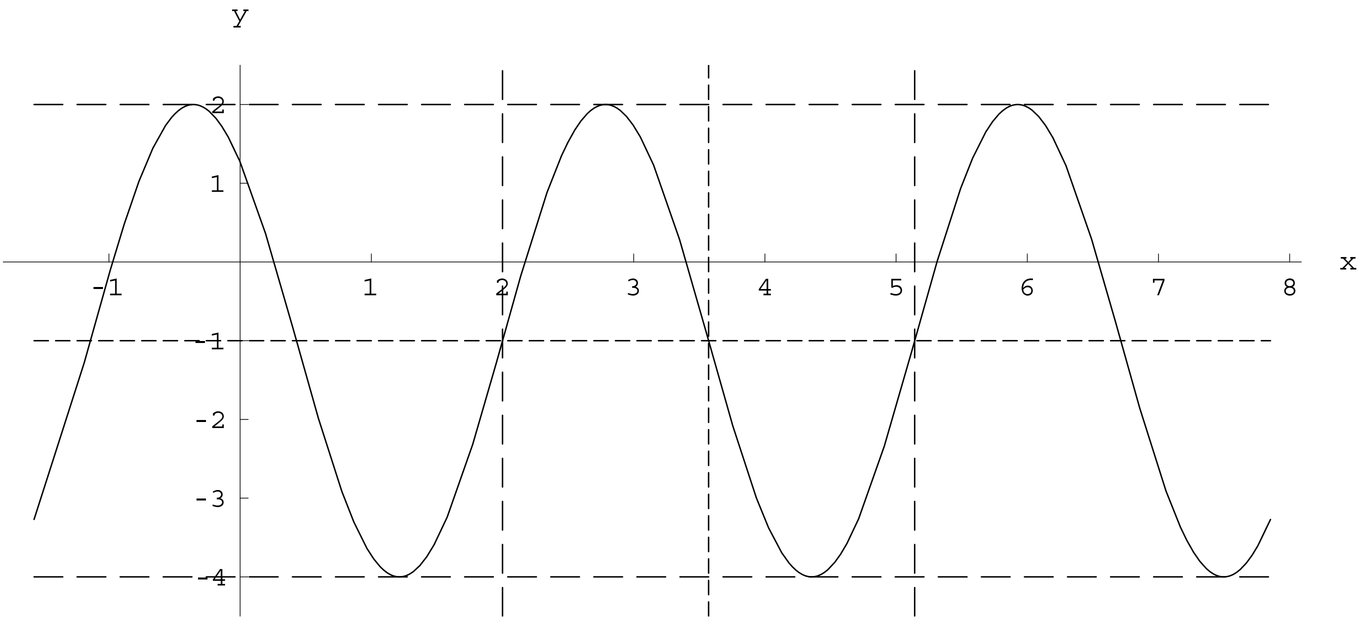
We have

amplitude 3

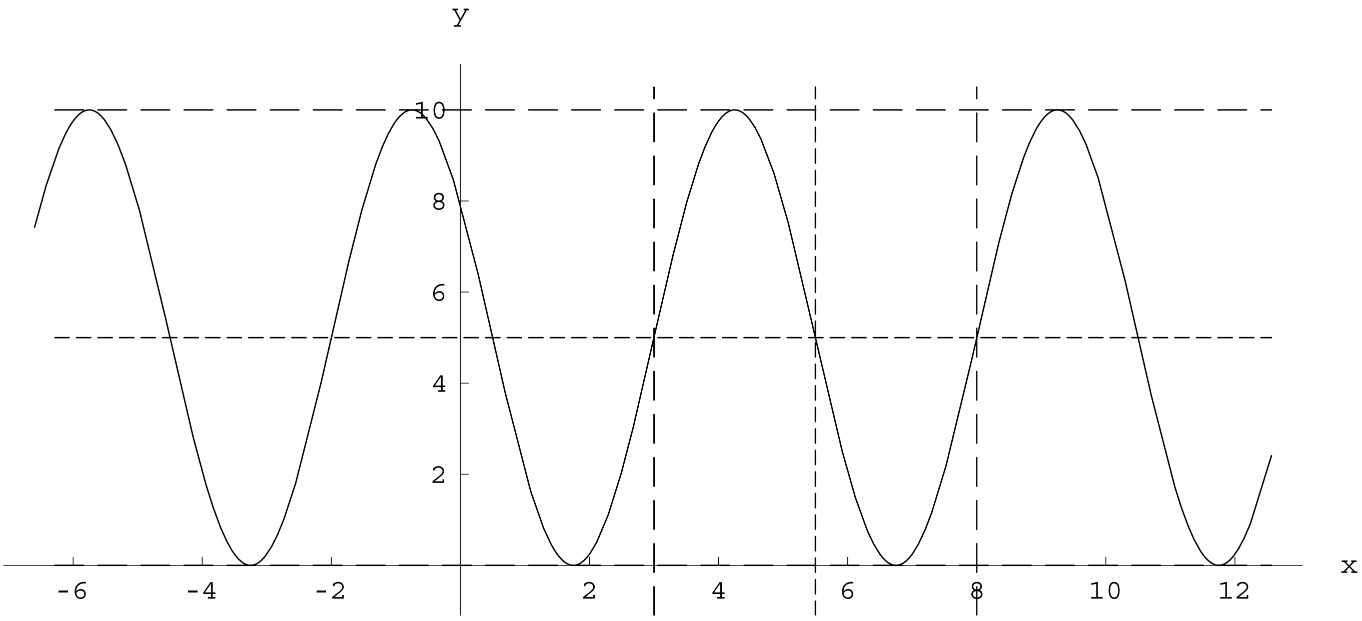
period π

mean value -1

and start 2.



Example



We see the vital statistics

amplitude 5

period 5

mean value 5

and start 3

and it's equation is

$$y = 5 + 5\sin\left[\tfrac{2\pi}{5}(x-3)\right].$$

Combining sinusoidal functions having the same period

[§*1.2.3 of Notes*]

If f and g are sinusoidal functions that have the *same period* then their sum is also sinusoidal. Adding two waves is often called “superposition”.

So, can we make the transformation

$$a \sin x + b \cos x \equiv^{5} R \sin(x + \alpha)$$

for some constants $R > 0$ and α, given a and b?

---

[5] This symbol means “equivalent”, and is stronger than “equals” (=). It means “true for ALL values”.

By a standard trigonometric formula [*Appendix 3 of Notes or Appendix D of Stewart*],

$$\begin{aligned} &R\sin(x+\alpha) \\ &\qquad = R\sin x\cos\alpha + R\cos x\sin\alpha \\ &\qquad = (R\cos\alpha)\sin x + (R\sin\alpha)\cos x. \end{aligned}$$

If this is equivalent to $a\sin x + b\cos x$, then we must have

$$\boxed{\begin{aligned} a &= R\cos\alpha \\ b &= R\sin\alpha. \end{aligned}} \qquad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

$$(1)^2 + (2)^2 \Rightarrow$$

$$\begin{aligned} a^2 + b^2 &= (R\cos\alpha)^2 + (R\sin\alpha)^2 \\ &= R^2(\cos^2\alpha + \sin^2\alpha) \\ &= R^2 \times 1 = R^2, \end{aligned}$$

so that, as $R > 0$,

$$\boxed{R = \sqrt{a^2 + b^2}}$$

and to determine α we use equations (1) and (2) to find α. We use both equations to find the α in the correct quadrant[6] which satisfies both equations simultaneously yielding a unique α within $0 \leq \alpha < 2\pi$.

Also commonly $-\pi < \alpha \leq \pi$ is used.

---

[6] Some students know this method from school as ASTC - "All Stations To Central"

Example

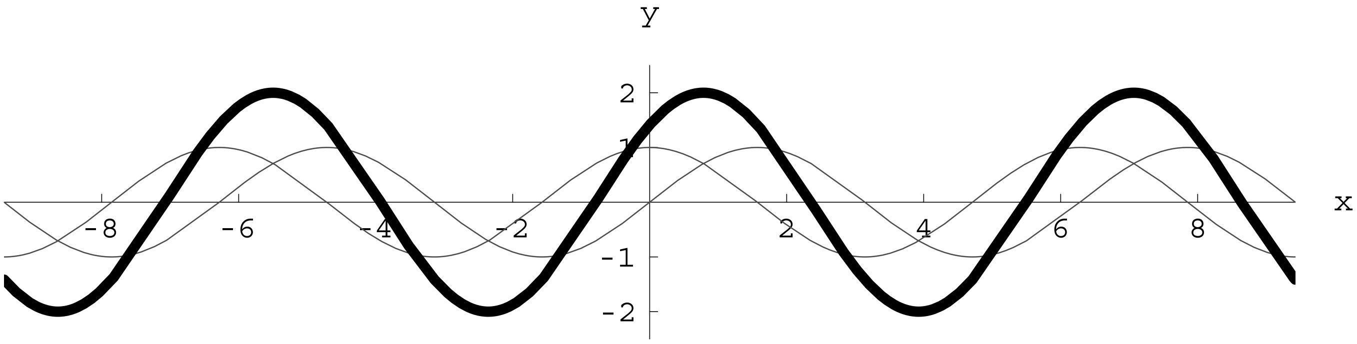
Write $\sin x + \cos x$ in the form $R\sin(x+\alpha)$ and hence sketch its graph.

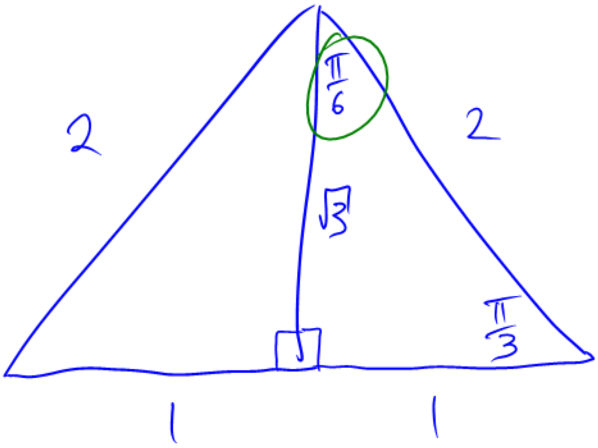
Now $\sin x + \cos x = 1 \cdot \sin x + 1 \cdot \cos x$ so we have $R = \sqrt{1^2+1^2} = \sqrt{2}$ and $\sin\alpha = \cos\alpha = \frac{1}{\sqrt{2}}$. Since $\sin\alpha$ and $\cos\alpha$ are both positive, α lies between 0 and $\frac{\pi}{2}$ (the first quadrant). In this interval the unique solution of $\sin\alpha = \frac{1}{\sqrt{2}}$ is $\alpha = \frac{\pi}{4}$.

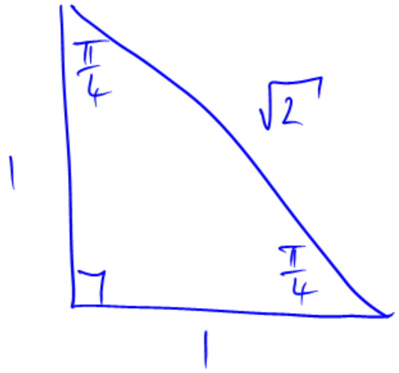
Need to know sin, cos & tan of $\frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{3}$ as well as integer multiples of $\frac{\pi}{2}$

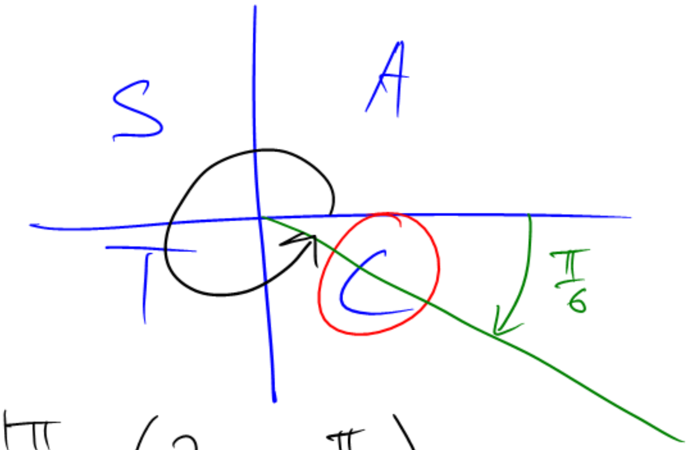
Hence

$$\sin x + \cos x = \sqrt{2}\sin\left(x + \tfrac{\pi}{4}\right).$$









$\therefore \alpha = -\frac{\pi}{6}$

or $\alpha = \frac{11\pi}{6} \quad \left(2\pi - \frac{\pi}{6}\right)$

Example

Find the greatest value of

$$4 + \tfrac{3\sqrt{3}}{2}\sin(2x) - \tfrac{3}{2}\cos(2x).$$

Consider

$$\tfrac{3\sqrt{3}}{2}\sin(2x) - \tfrac{3}{2}\cos(2x).$$

We have

$$R = \sqrt{\left(\tfrac{3\sqrt{3}}{2}\right)^2 + \left(\tfrac{-3}{2}\right)^2} = \sqrt{\tfrac{27}{4} + \tfrac{9}{4}} = 3.$$

Also $\cos\alpha = \frac{\sqrt{3}}{2}$ and $\sin\alpha = -\frac{1}{2}$. Since $\sin\alpha$ is negative and $\cos\alpha$ is positive, α lies between $\frac{3\pi}{2}$ and 2π (the 4th quadrant).

In this interval, the unique solution of $\sin\alpha = -\frac{1}{2}$ is $\alpha = \frac{11\pi}{6}$. Hence

$$\frac{3\sqrt{3}}{2}\sin(2x) - \frac{3}{2}\cos(2x) = 3\sin\left(2x + \frac{11\pi}{6}\right).$$

Hence the greatest value[7] of

$$4 + \frac{3\sqrt{3}}{2}\sin(2x) - \frac{3}{2}\cos(2x)$$

is $4 + 3 = 7$.

---

[7] You may realise that finding α is a waste of time in this problem, but we'll need it for the next example.

Secret Formula!

Now there is a "secret formula"[8] which is a real shortcut for finding α. One may originally think that

$$\begin{aligned}(2) \div (1) &\Rightarrow \tan\alpha = \frac{b}{a} \\ &\Rightarrow \alpha = \tan^{-1}\left(\frac{b}{a}\right)\end{aligned}$$

but this turns out to only work if a and b are both positive. The slight adaption of this formula

$$\boxed{\alpha = 2\tan^{-1}\left(\tfrac{b}{a+R}\right)}$$

---

[8] Okay so admittedly it's not so secret. You can find it at `http://en.wikipedia.org/wiki/Argument_(complex_analysis)`

is guaranteed to work in all cases[9].

Let's check this with the previous example.

We have $a = \frac{3\sqrt{3}}{2}, b = -\frac{3}{2}, R = 3$ so

$$\alpha = 2\tan^{-1}\left\{\frac{-\frac{3}{2}}{\frac{3\sqrt{3}}{2} + 3}\right\} = -\frac{\pi}{6}$$

which is equivalent[10] to $\frac{11\pi}{6}$, so it does indeed work!

---

[9] In this course you may choose to use either the secret formula or the ASTC method.

[10] Because $-\frac{\pi}{6} + 2\pi = \frac{11\pi}{6}$.