Data Scaling

[§*1.3 of the Notes; in Stewart: start of* §*1.2 has some examples of linear models, and a revision of straight line geometry in Appendix B*]

In life we often try to predict things. We build a "model" that describes some given data, and use the model to predict data that we don't yet have.

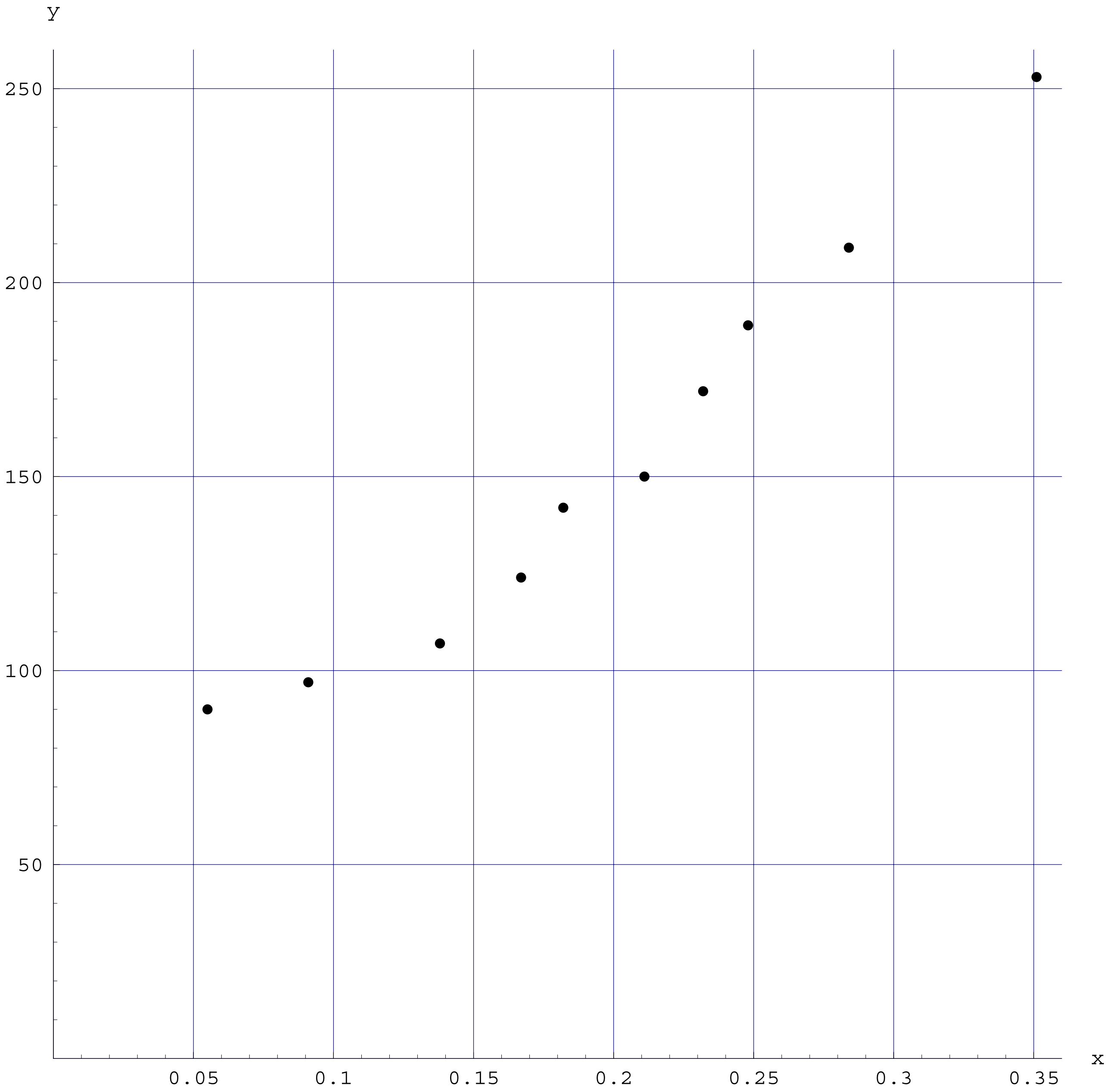
The idea is very simple. Straight lines are easy to recognise – you just put a ruler against the data points. Nothing else is as easy to recognise.

Example

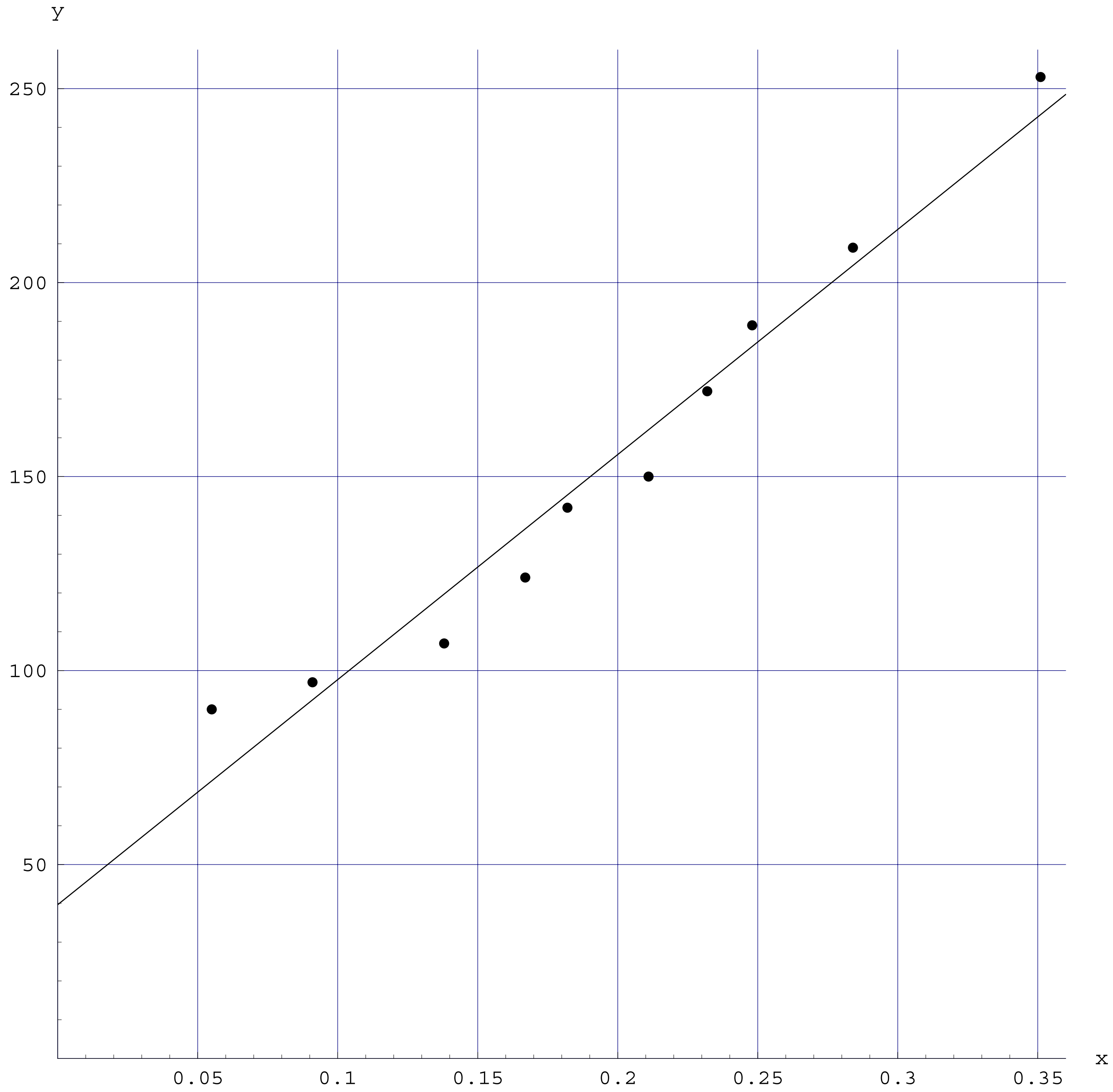
Suppose you have some experimental data:

x	0.055	0.091	0.138	0.167	0.182
y	90	97	107	124	142
x	0.211	0.232	0.248	0.284	0.351
y	150	172	189	209	253

You could plot the data points in the xy-plane: see that they (roughly) lie on a straight line (check using a ruler), find the slope and y-intercept, and write down a formula for the line. Use it to predict other data points.



y
250
200
150
100
50
0.05
0.1
0.15
0.2
0.25
0.3
0.35
x


y
250
200
150
100
50
0.05
0.1
0.15
0.2
0.25
0.3
0.35
x

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope[1] is $m = \dfrac{243 - 40}{0.35 - 0} = 580.$

b

The y-intercept is 40. So the equation of the line which fits the data is

$y = mx + b$

$$y = 580x + 40$$

This gives a *model* for the data $y = f(x) = 580x + 40$ which could be used for predicting y values at other values of x.

[1]Notice that this data was obtained directly from the line of best fit itself, NOT the data!

If the data does not behave nicely and follow such a linear law, the next best thing to do is to try and find transformations of the x and y data which make the data fit a straight line. Then you can write down the linear relationship and invert the transformation to get a good model.

Now we take a detour before coming back to the theme above.

Proportionality [§*1.3.1 of Notes*]

produces linear models

We say that y is proportional to x and write $y \propto x$ when $y = kx$, where k is a constant (called *the constant of proportionality*).

Example

The volume of paint needed to cover a wall is proportional to the area of the wall.

If a particular wall needs x litres of paint a similar wall with double the area needs to $2x$ litres and one with three times the area needs $3x$ litres.

$$V \propto A \Rightarrow V = kA$$

where A is the area of the wall in square metres, V the volume of paint in litres and k is the constant of proportionality.

k does not depend on A and V, but it will vary from one paint to another and perhaps with the porosity of the wall being painted.

Suppose that for a given paint and a given wall surface $k = \frac{1}{25}$, when the wall area is measured in square metres and the volume of paint is measured in litres.

Then $V = \frac{1}{25}A$ and a wall of area at 10 square metres takes $\frac{1}{25} \times 10 = 0.4$ litres of paint.

Linear Functions [*§1.3.2 of Notes, Appendix B of Stewart*]

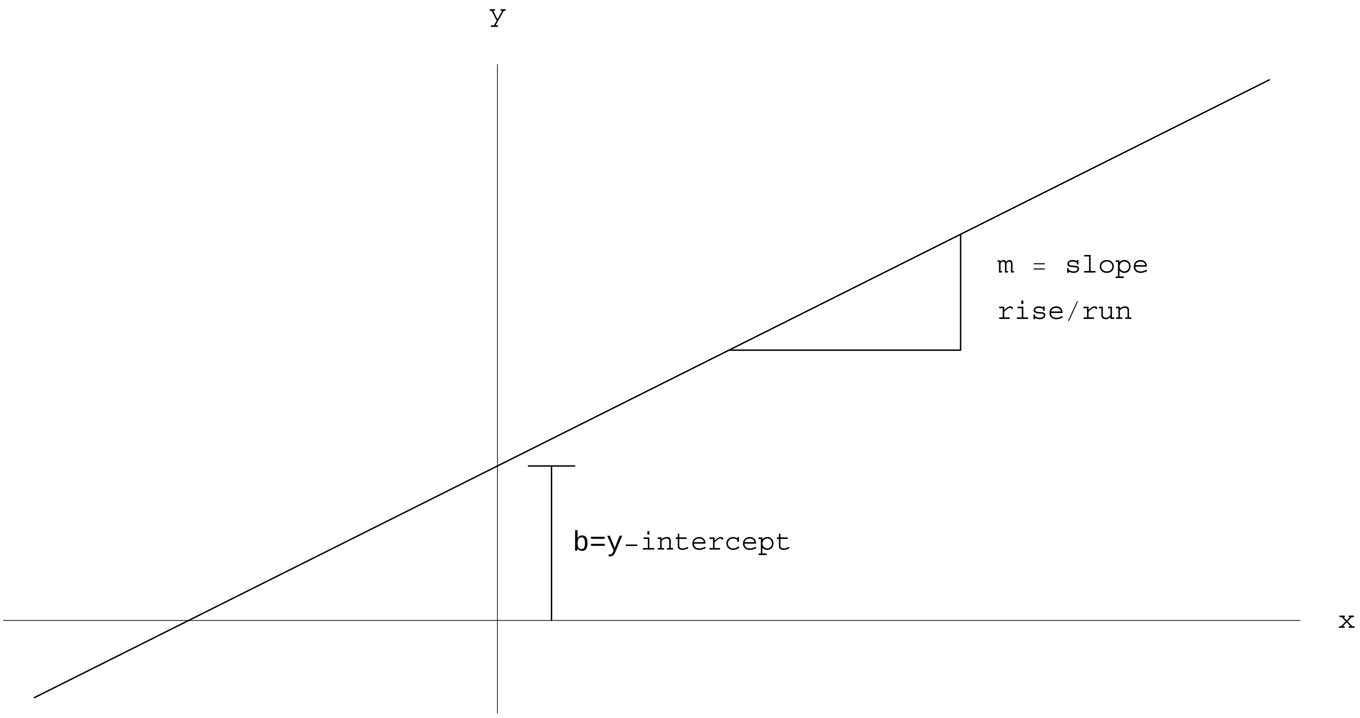
Two variables, x and y are linearly related if there are constants A, B and C such that $Ax + By + C = 0$.

Linearly related variables have straight-line graphs.

I remind you that the equation[2] of the line with slope m and y-intercept b is

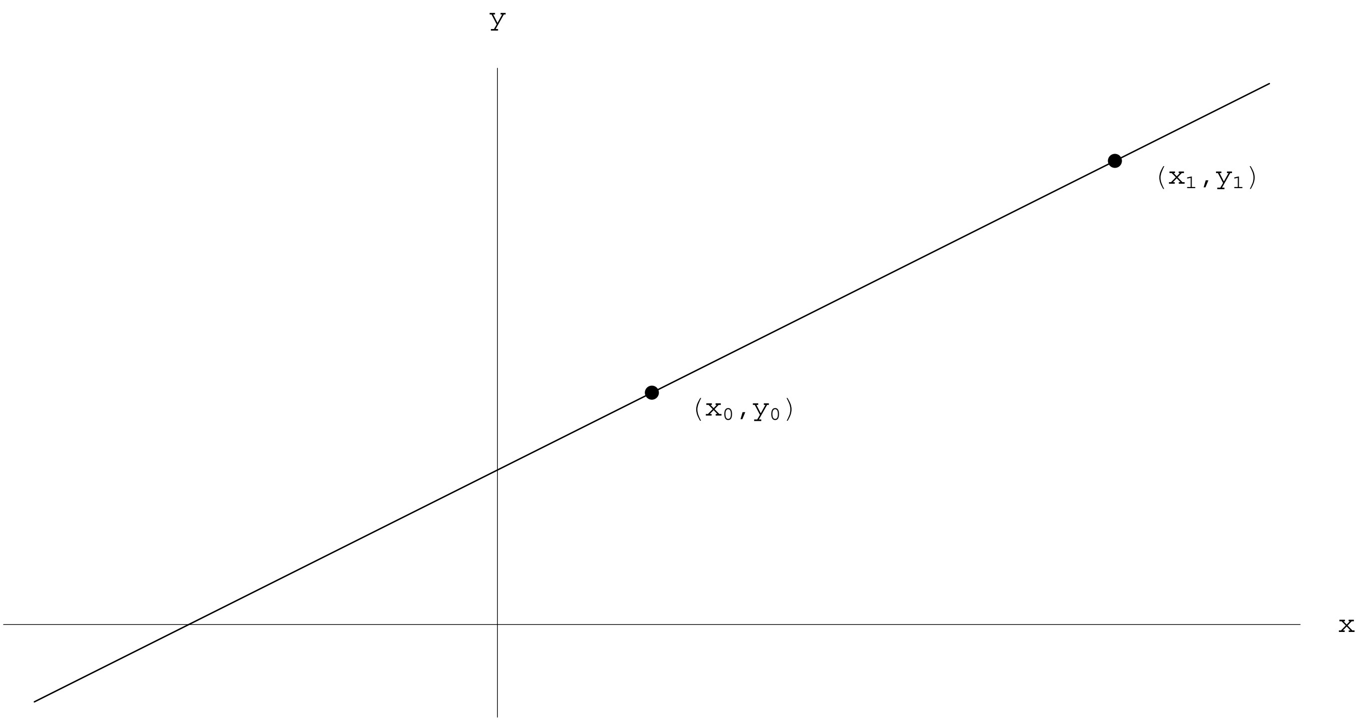
$$\boxed{y = mx + b.}$$

[2]Often called the "gradient-intercept" form.


y
m = slope
rise/run
b=y-intercept
x

I remind you that the equation[3] of the line joining two points (x_0, y_0) and (x_1, y_1) is

$$\boxed{\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0}}$$



[3]Often called the "two point" formula.

Example

Find the equation of the line joining $(1,2)$ and $(3,7)$.

The equation is

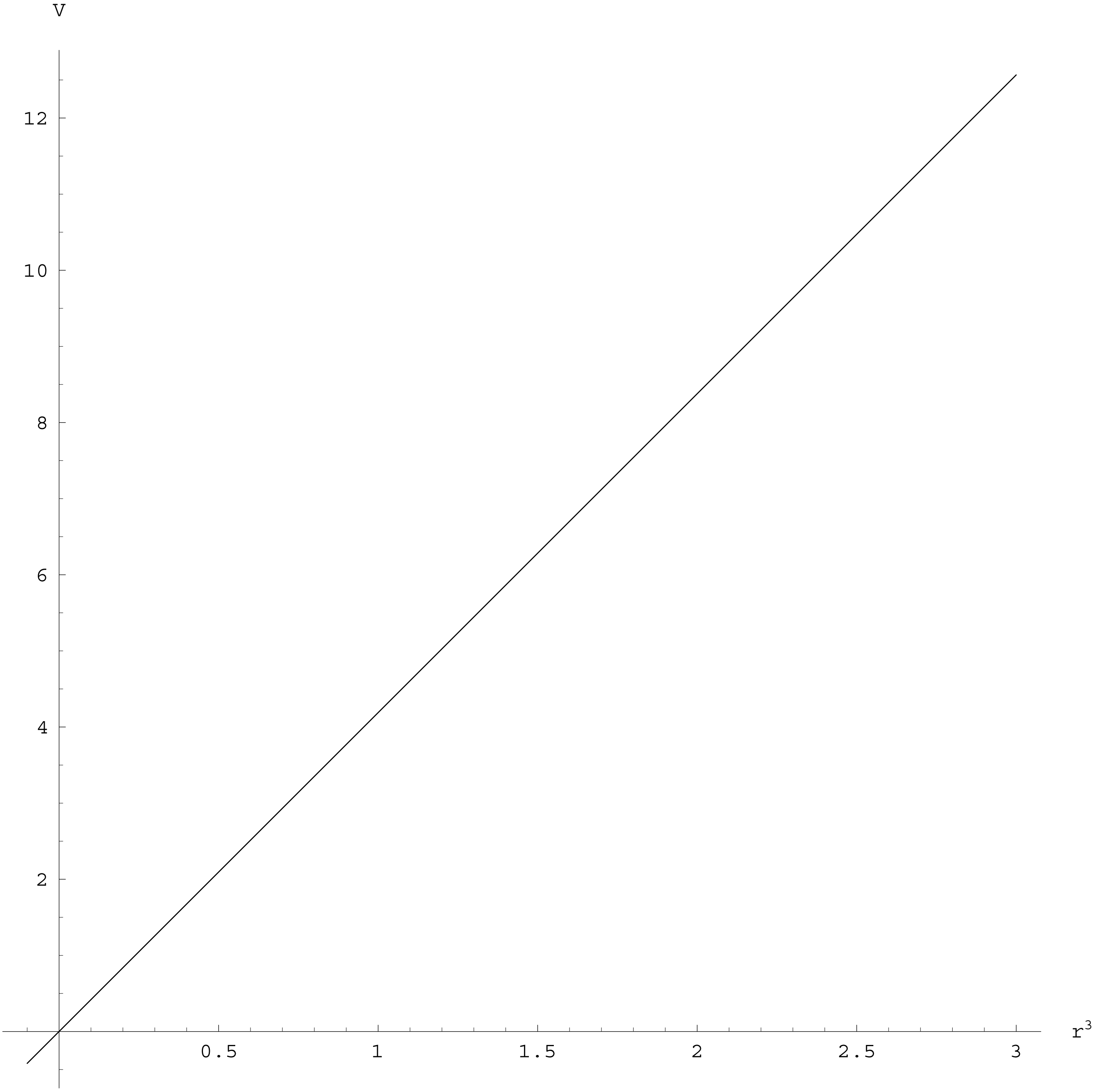
$$\begin{aligned} \frac{y-2}{x-1} &= \frac{7-2}{3-1} \\ \Leftrightarrow y-2 &= \tfrac{5}{2}(x-1) \\ \Leftrightarrow 2y-4 &= 5x-5 \\ \Leftrightarrow 5x-2y-1 &= 0 \\ \Leftrightarrow y &= \tfrac{1}{2}(5x-1) \end{aligned}$$

Example

The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

The volume of a sphere is proportional to the cube of its radius (so $V \propto r^3$).

If you plot the volume of a sphere against the cube of its radius you get a straight line through the origin.



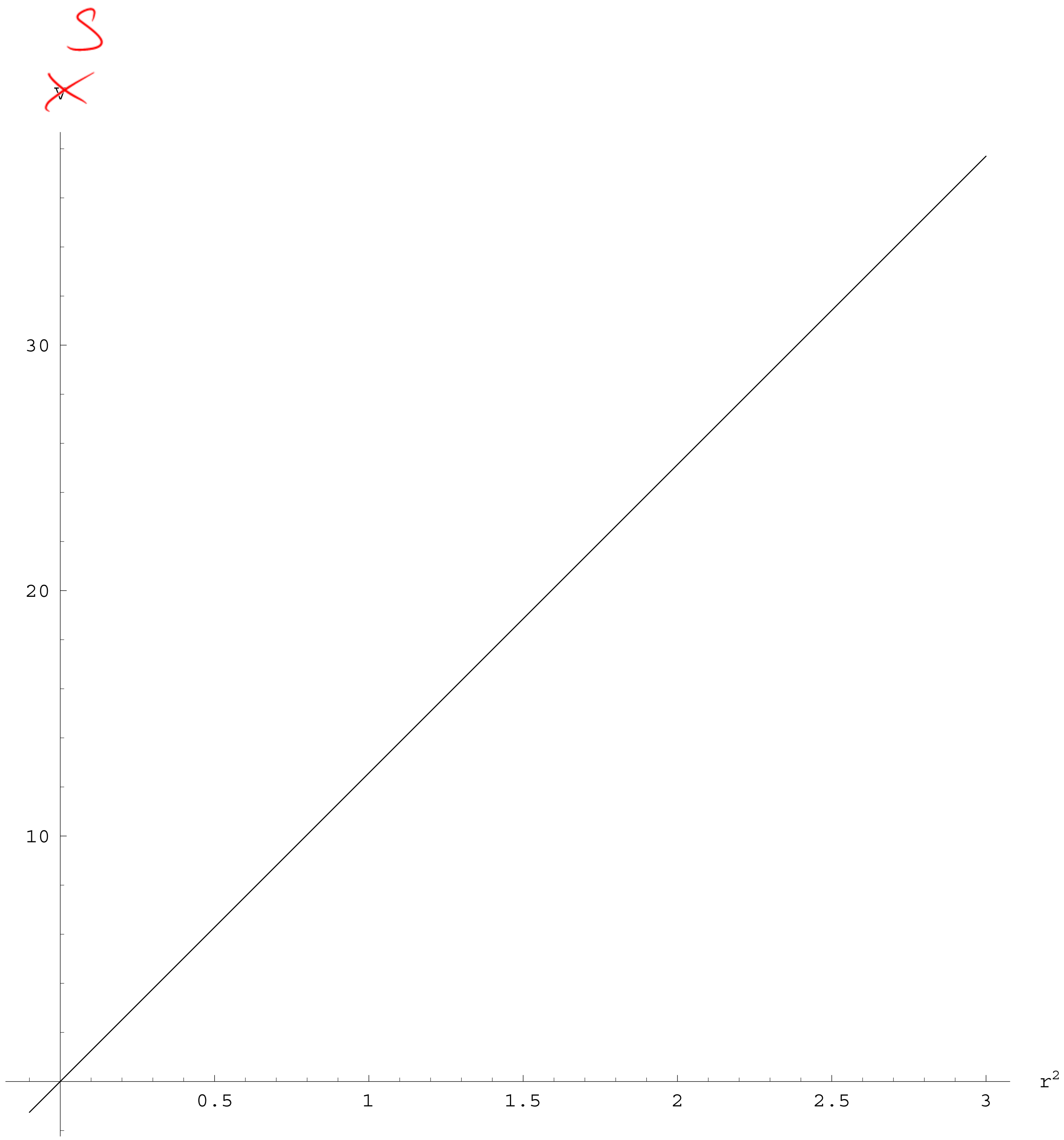
V
12
10
8
6
4
2
0.5
1
1.5
2
2.5
3
r3

Example

The surface area of a sphere of radius r is $S = 4\pi r^2$.

The surface area of a sphere is proportional to the square of its radius (so $S \propto r^2$).

If you plot the surface area of a sphere against the square of its radius you get a straight line through the origin.

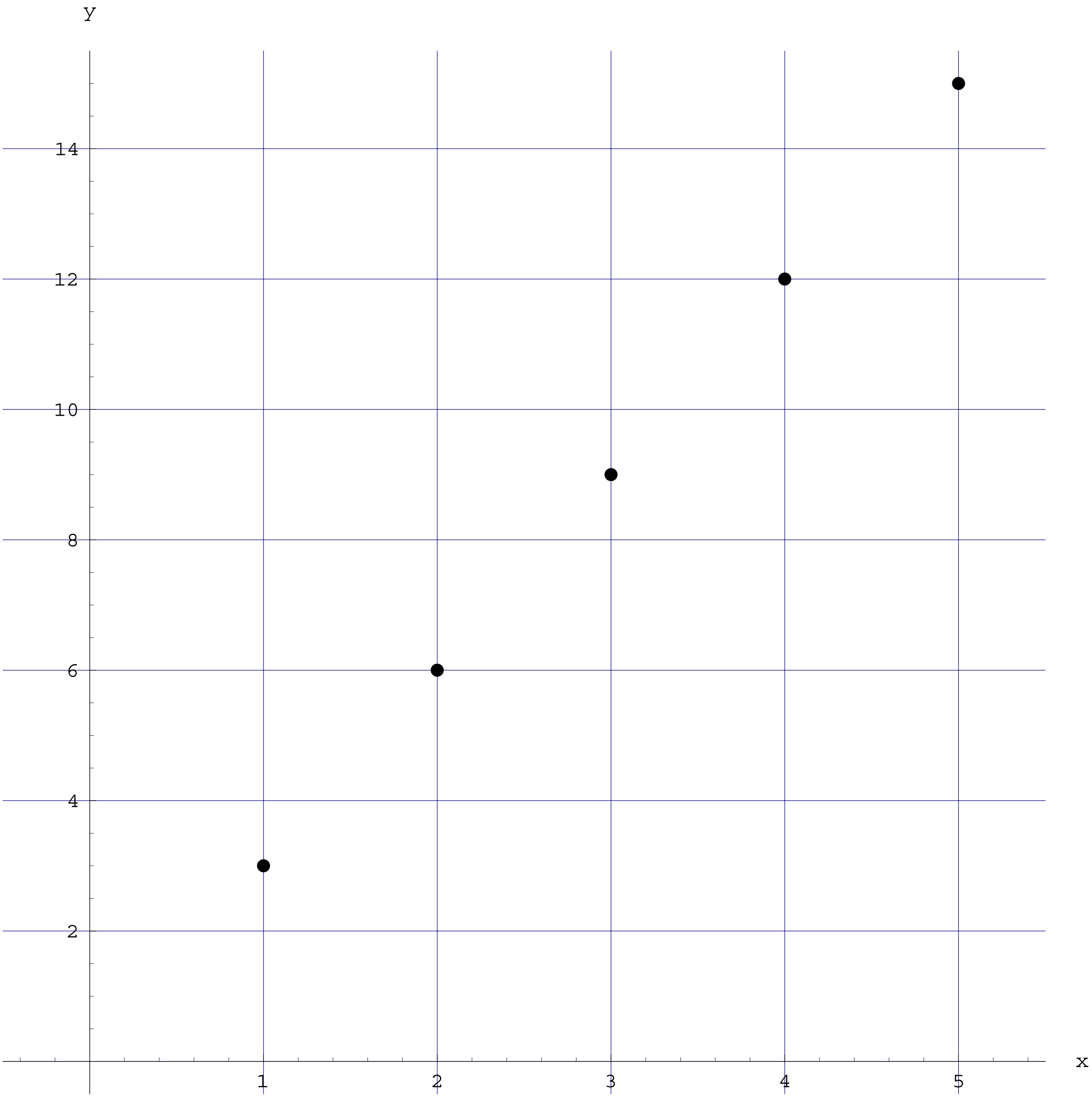

S
v
30
20
10
0.5
1
1.5
2
2.5
3
r^2

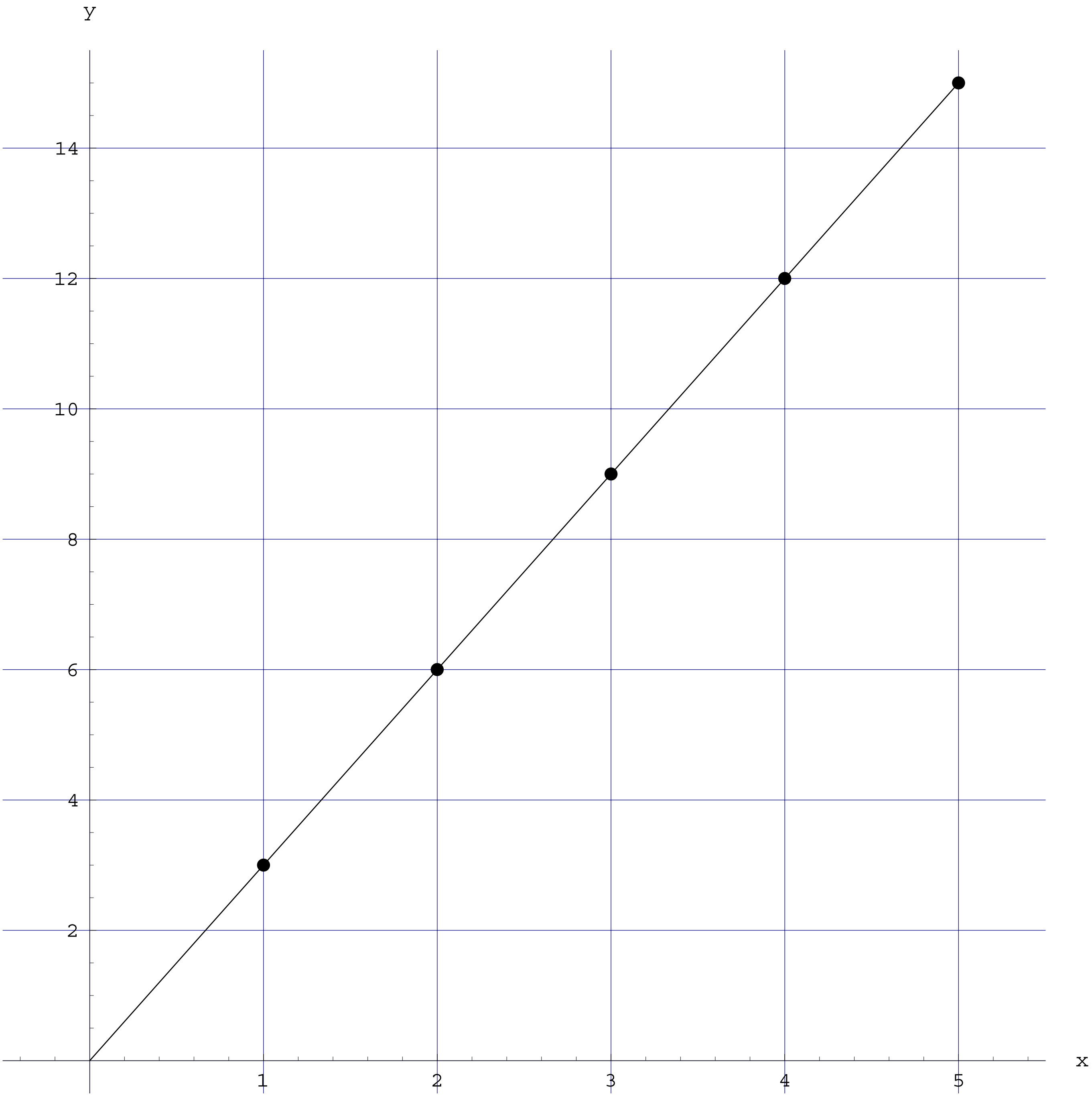
Let us move back to the situation, described above, where we have experimental data which we wish to model. That is we want to find a formula which fits the data and can be used for forecasting.

Sometimes the relationship is clear. If you to plot the data and get a straight-line you have a linear relationship and you can write down a formula easily.

Example

x	1	2	3	4	5
y	3	6	9	12	15



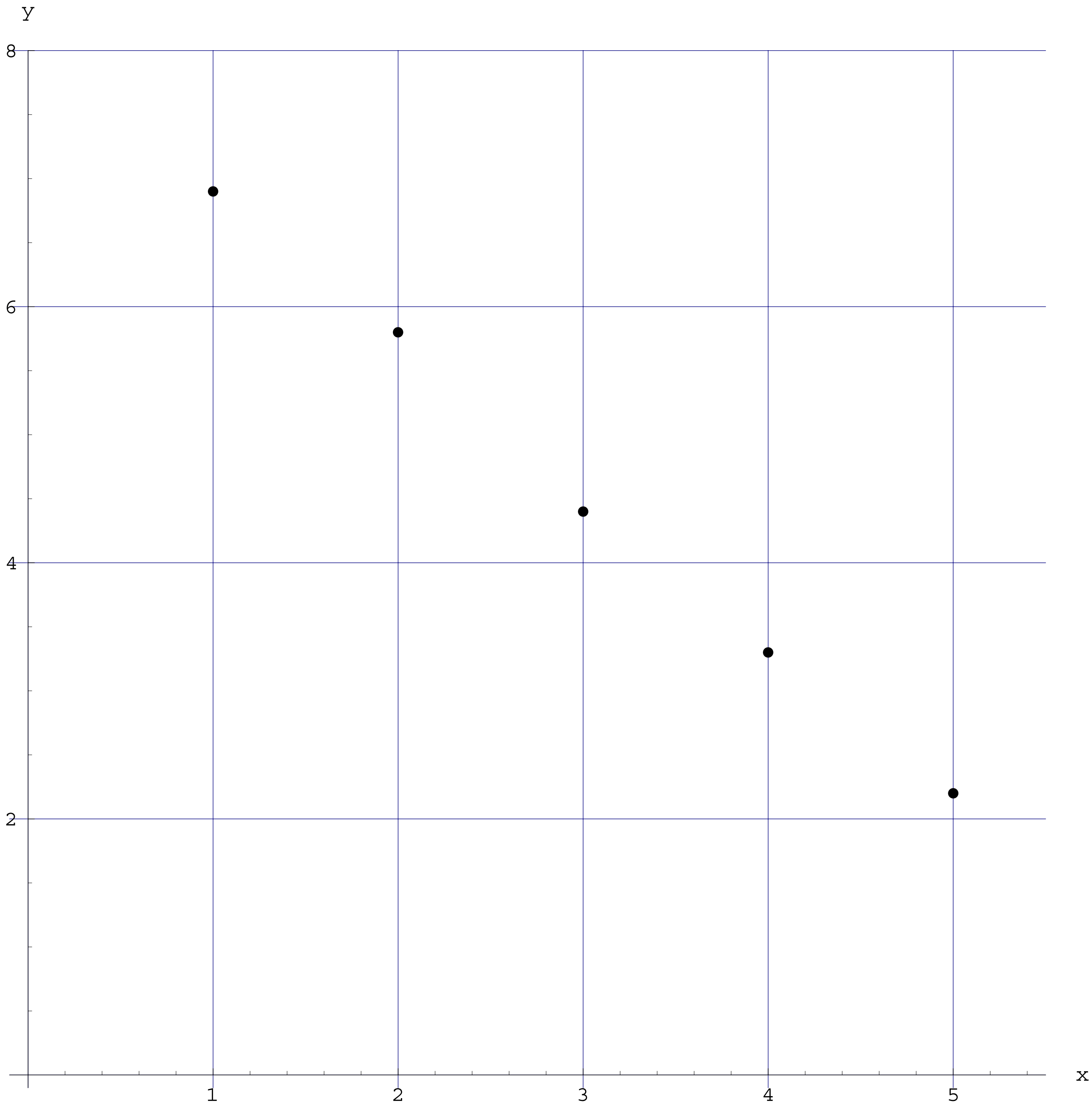


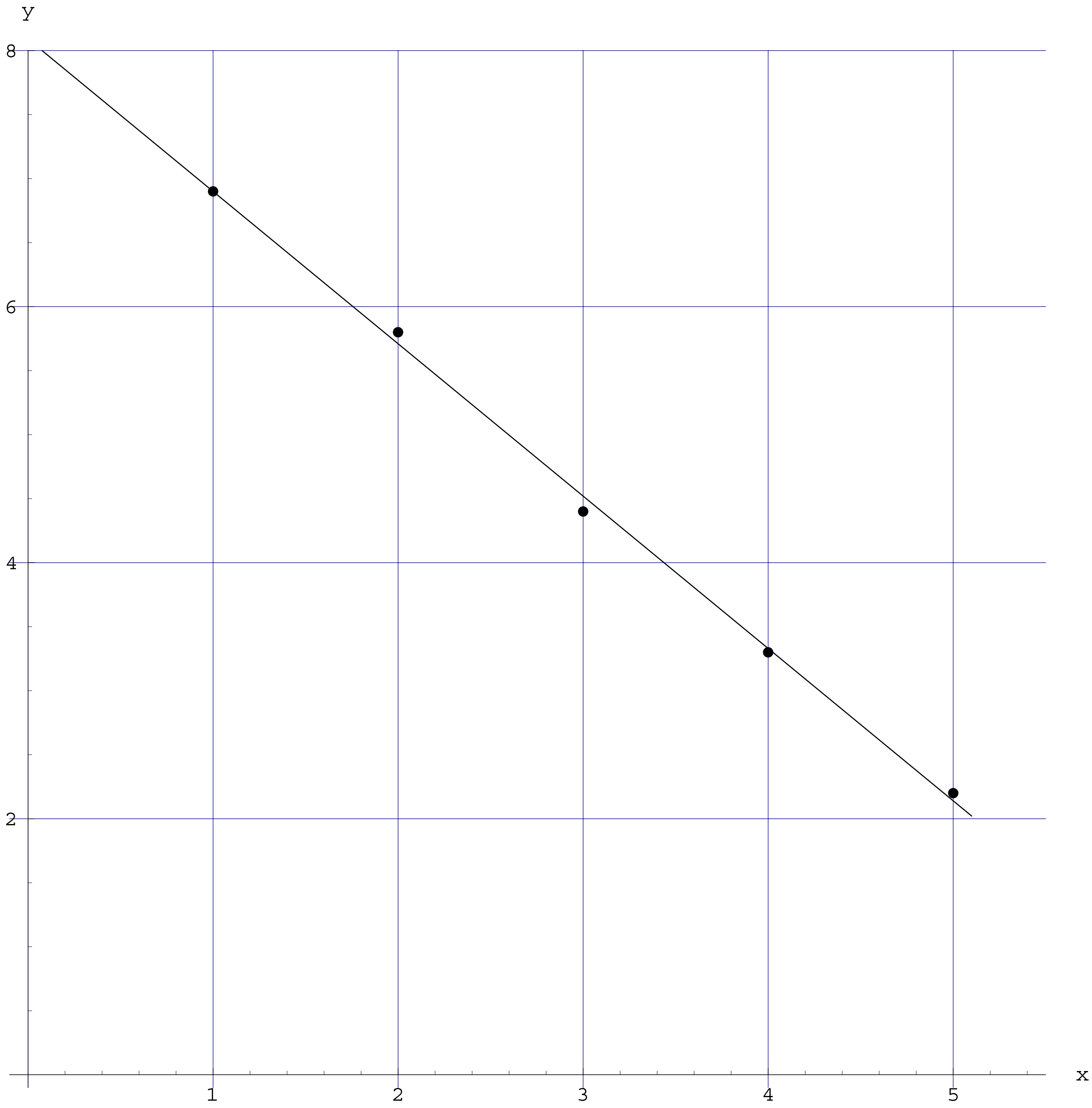
Model: $y = 3x$

Example

A researcher suspects that the following data can be explained by a linear relationship between x and y. Show that this is so and find the relationship.

x	1	2	3	4	5
y	6.9	5.8	4.4	3.3	2.2


y
8
6
4
2
1
2
3
4
5
x


y
8
6
4
2
1
2
3
4
5
x

The points are very close to falling on a straight line. So a linear model will fit the data well.

Now we need two points that sit on the line of best fit. The most accurate model would come from choosing the farthest endpoints of the line. However the co-ordinates would be hard to read given the scale of the axes on that graph, so we will approximate the endpoints of the line of best fit with the first and last data points, since they are close enough to the line anyway[4].

[4] In real life you would have drawn a good scale and could accurately measure the line's endpoints with a ruler or computer.

Taking the line joining $(1, 6.9)$ and $(5, 2.2)$, we get

$$\frac{y - 6.9}{x - 1} = \frac{2.2 - 6.9}{5 - 1} = -1.2.$$

So the formula which models the data is $y = -1.2x + 8.1$.

This formula models the data reasonably well.

Example

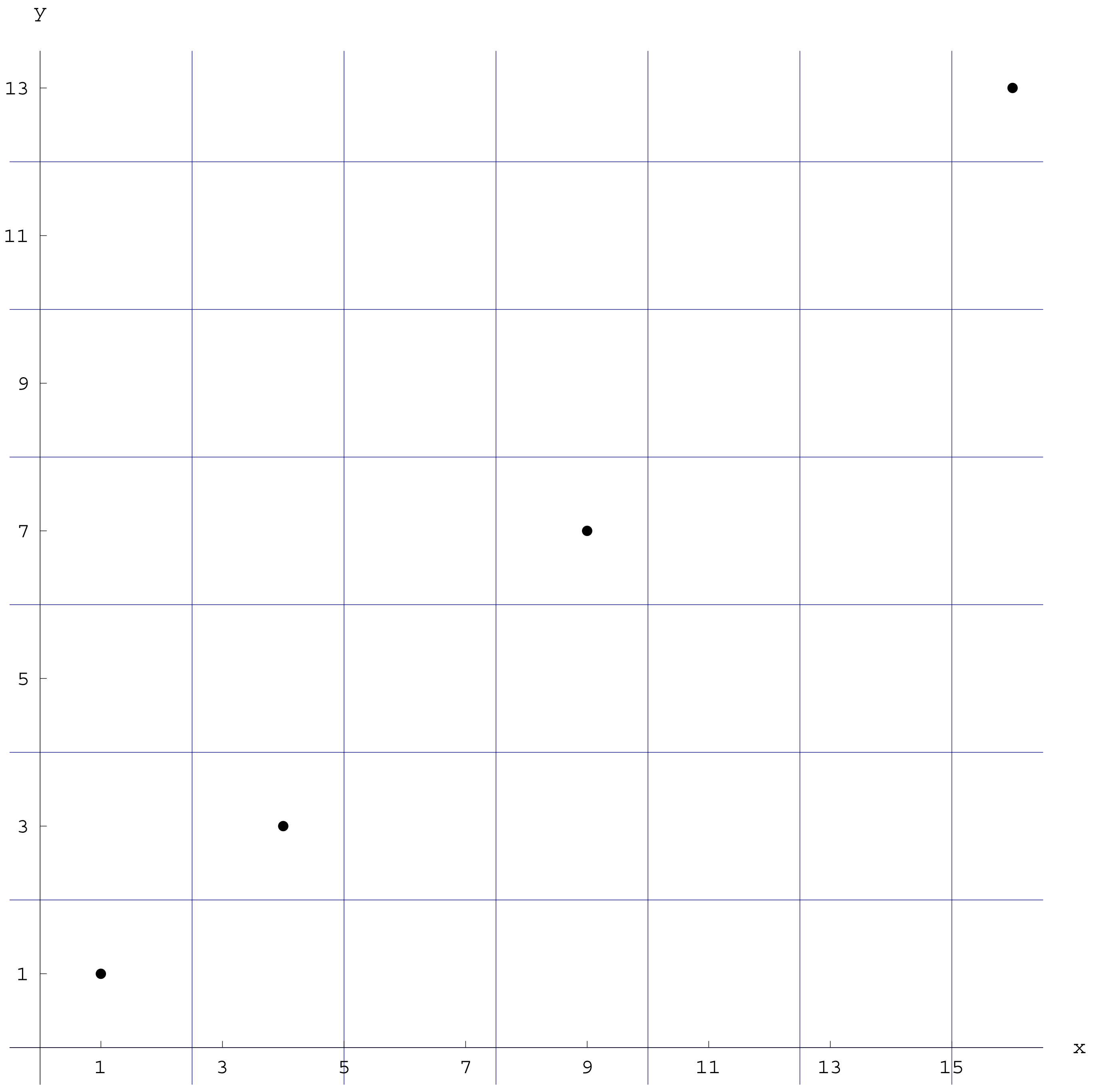
A researcher is hoping to show that a variable y is proportional to the square of the variable x. She has the following data.

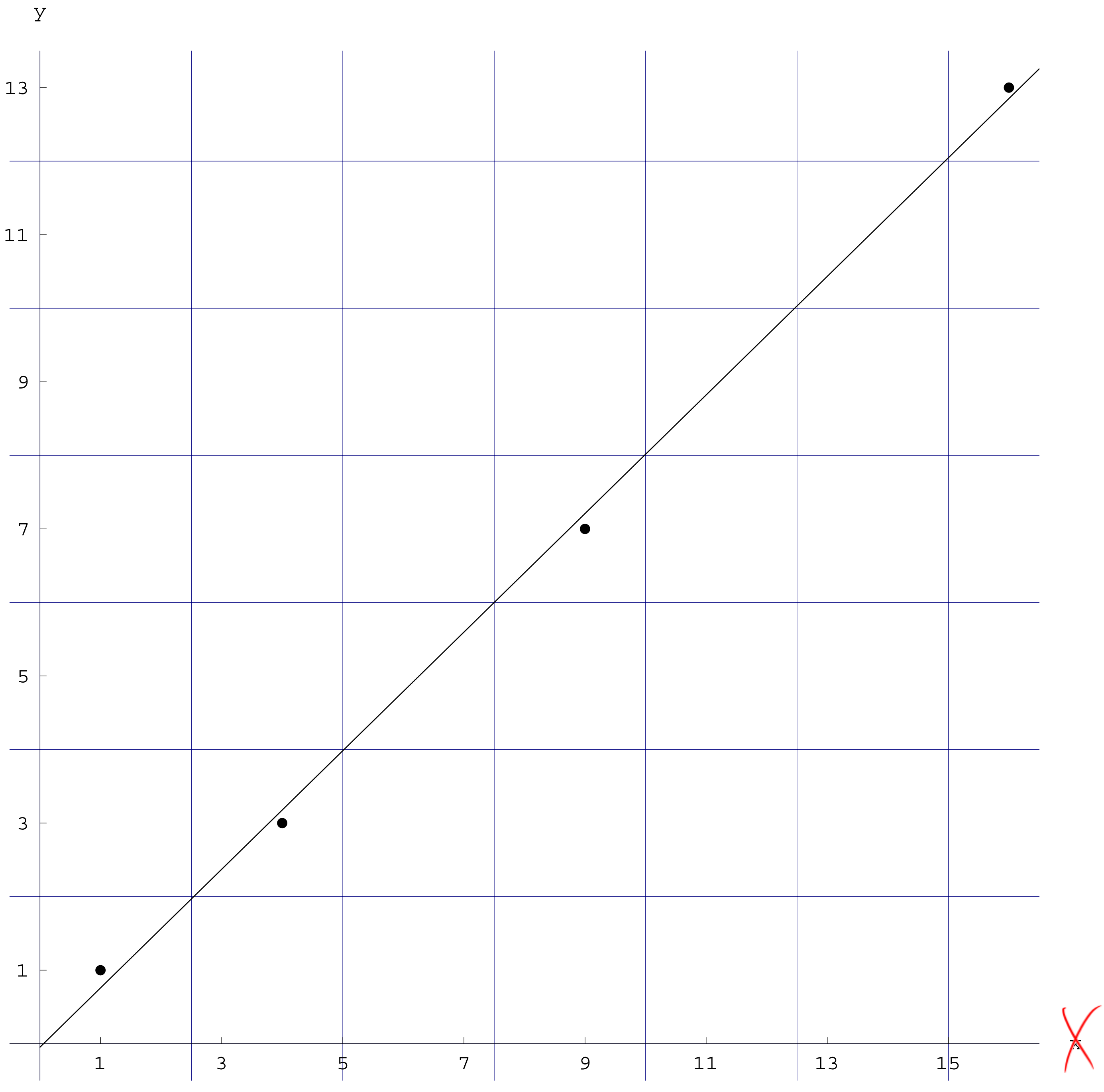
x	1	2	3	4
y	1	3	7	13

Let $X = x^2$. So that

x	1	2	3	4
X	1	4	9	16
y	1	3	7	13

We plot y against X.


y
13
11
9
7
5
3
1
x
1
3
5
7
9
11
13
15


y
13
11
9
7
5
3
1
1
3
5
7
9
11
13
15

The points are very close to falling on a straight line. So a linear model (for y and X) will fit the data well.

Taking[5] the line joining $(0, 0)$ and $(16, 13)$, we get

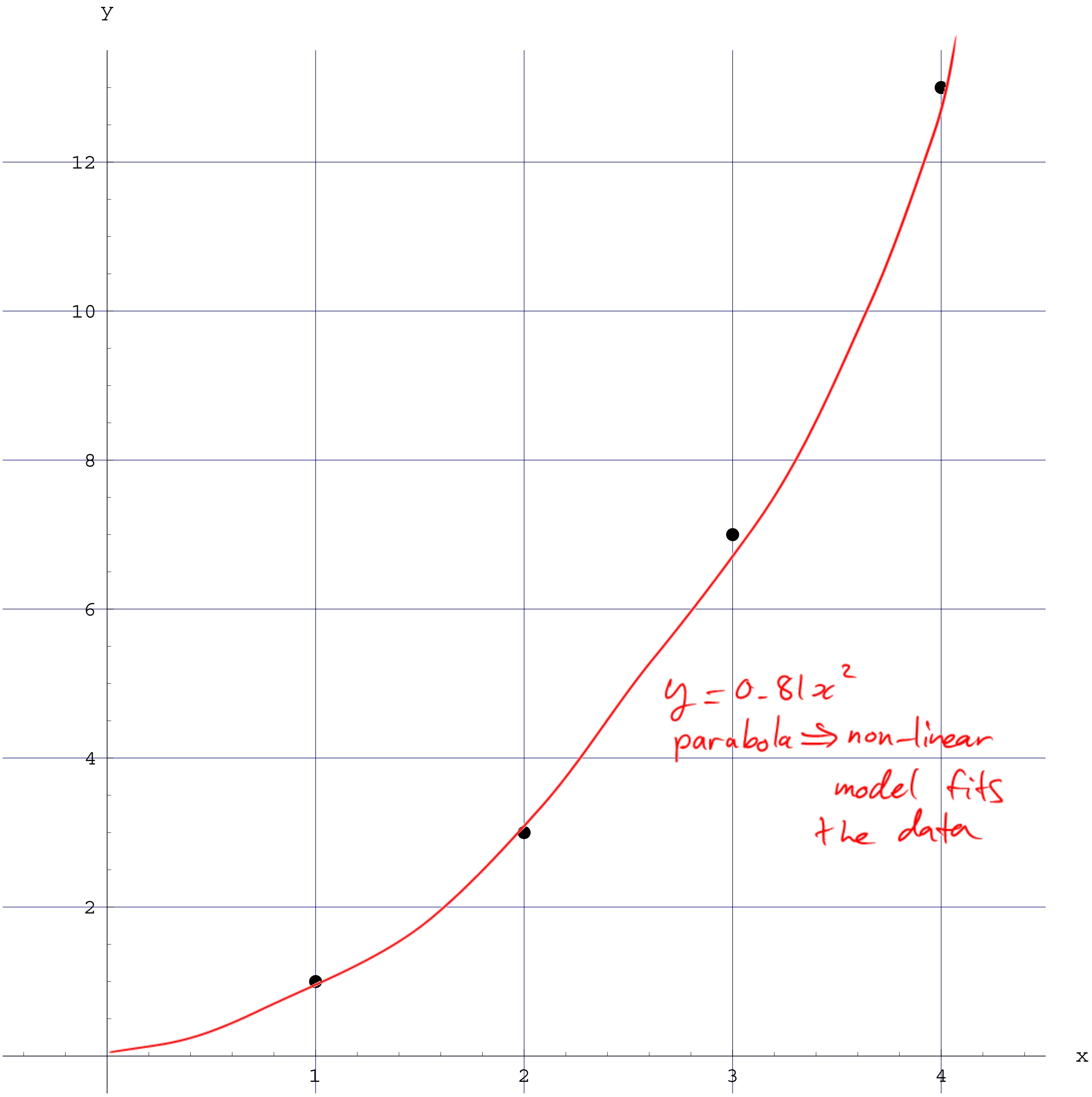
$$\frac{y}{X} = \frac{13}{16}.$$

So a formula which models the data is $y = 0.81X = 0.81x^2$.

This formula models the data reasonably well.

[5] Again we're limited by the accuracy of this scale so we have no other choice.

What we have just done is a good example of “scaling” or “transforming” data. If you plot y against x you get the following plot and the relationship is not all obvious. Plotting y against $X = x^2$ gives a straight line which is unmistakable.


y
12
10
8
6
4
2
1
2
3
4
x
y = 0.81x^2
parabola ⇒ non-linear
model fits
the data

Example

Check that Kepler's third law of planetary motion, $T^2 \propto a^3$, is reasonable. Here a is the mean distance of a planet from the Sun (measured in multiples of the Earth's mean distance from the Sun) and T is the length of the planet's[6] year (measured in terms of the Earth's year) called the *period* of the orbit.

We have

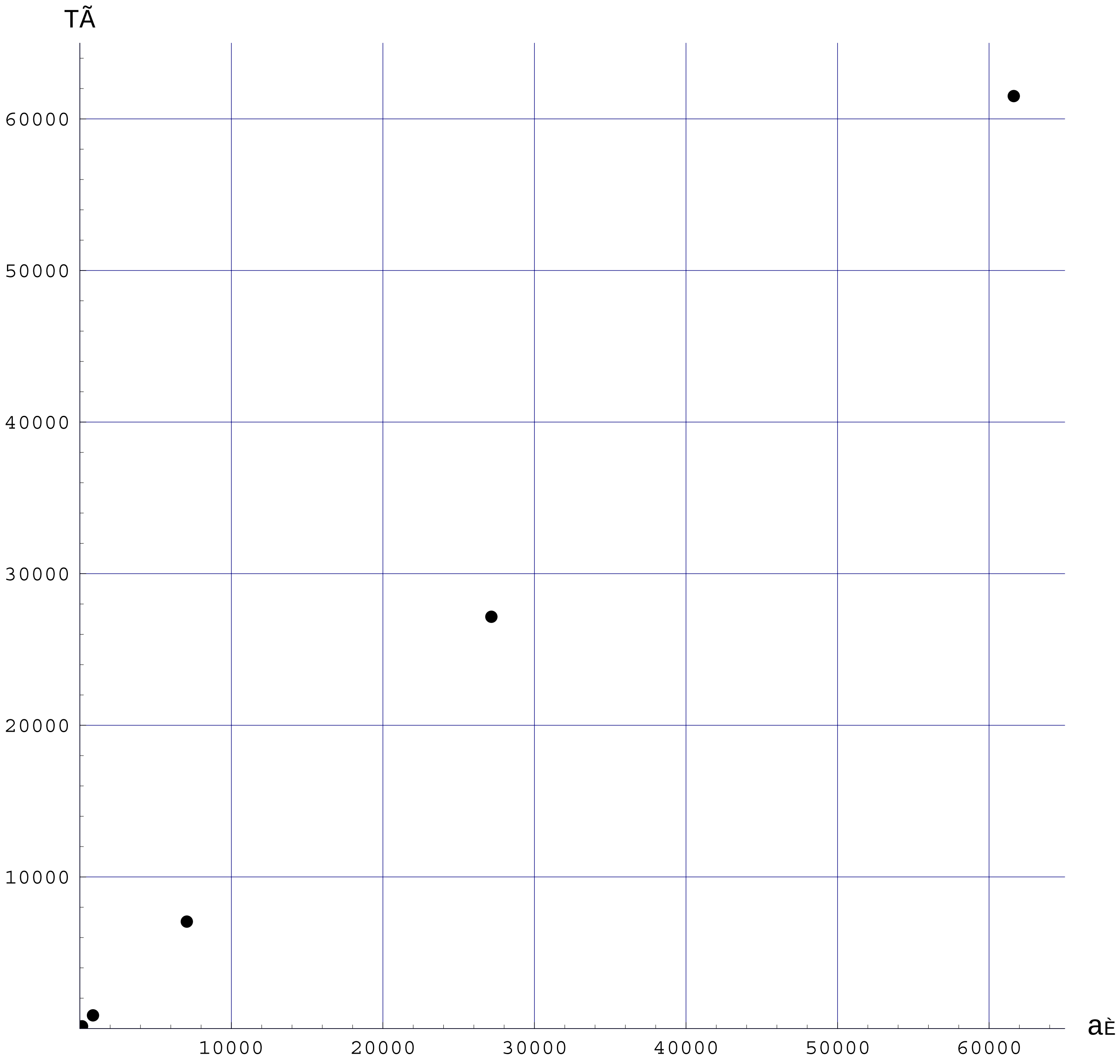
[6] Although Pluto is no longer considered a planet, it is still a satellite of the Sun so should still obey Kepler's Law (but will have the largest error/deviation from this law).

Planet	M	V	E	M	J
a	0.387	0.723	1.000	1.524	5.204
T	0.241	0.615	1.000	1.881	11.86
Planet	S	U	N	P	
a	9.539	19.18	30.06	39.5	
T	29.46	84.0	164.8	248	

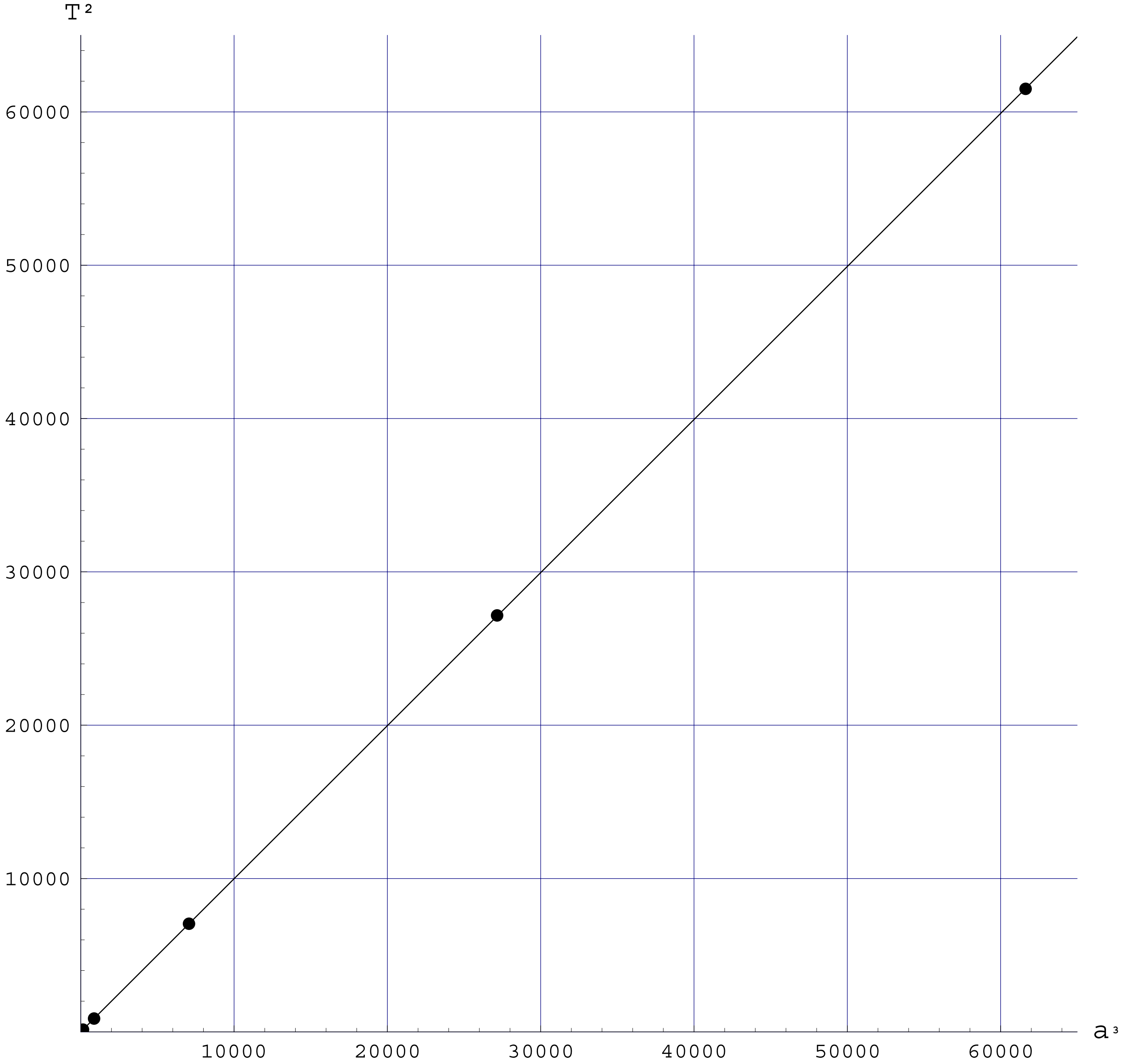
First make a table of T^2 against a^3.

a^3	0.058	0.378	1.000	3.540	141
T^2	0.058	0.378	1.000	3.538	141
a^3	868	7056	27162	61629	
T^2	868	7056	27159	61504	

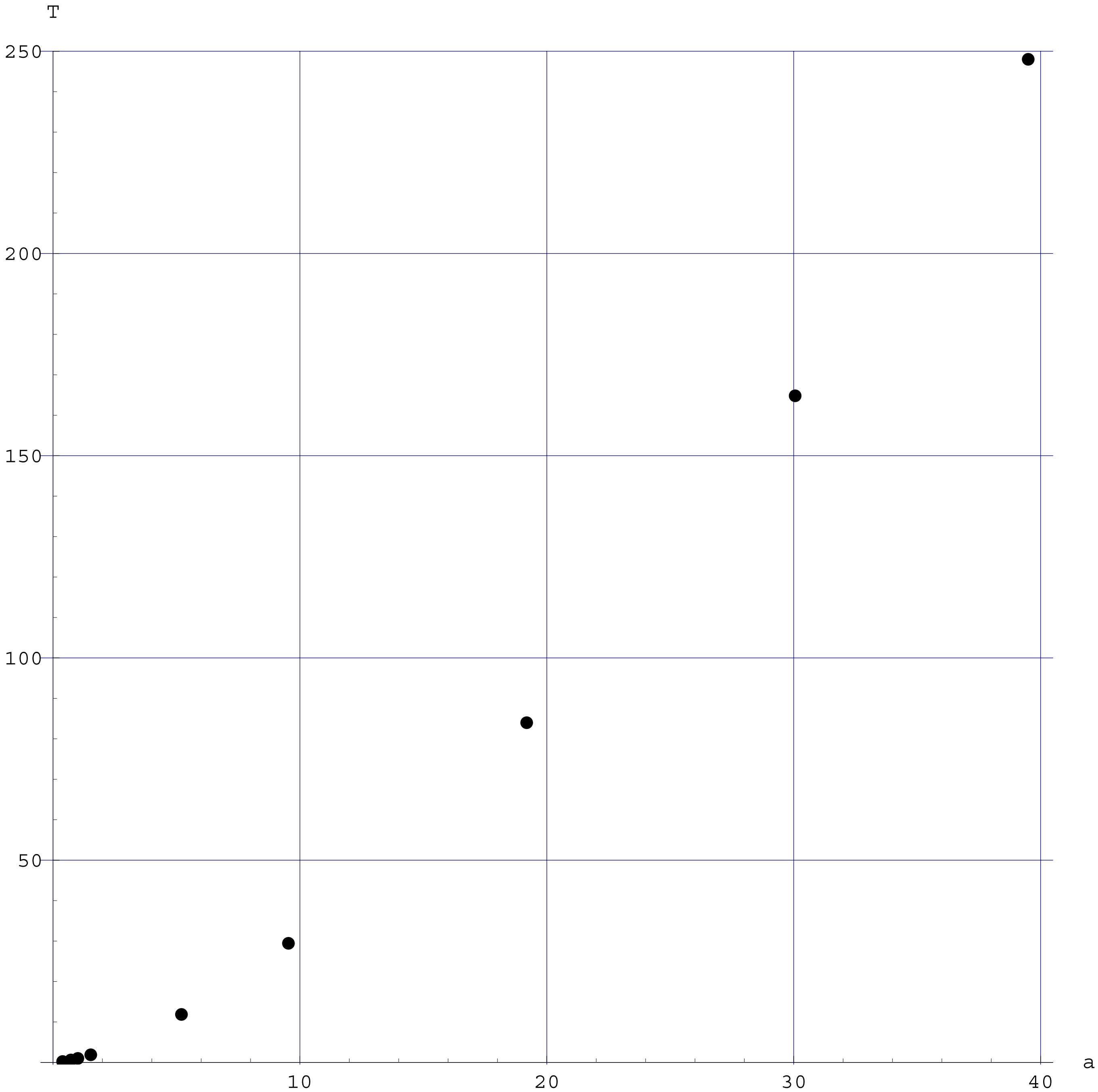
It is clear, without plotting, that $T^2 = a^3$ (or $T = a^{\frac{3}{2}}$) gives a very good model indeed.



TÃ
60000
50000
40000
30000
20000
10000
10000
20000
30000
40000
50000
60000
aÈ



It is clear that $T^2 = a^3$
(or $T = a^{\frac{3}{2}}$).



Without scaling.

Logarithmic Scaling

[§*1.3.3 of Notes*, §§*7.2-7.4 of Stewart*]

Many natural laws fall into one of the following two forms:

$$\boxed{\begin{array}{ll} y = Ae^{kx} & \textit{exponential} \\ y = ax^n & \textit{allometric} \end{array}}$$

Both of these relationships turn out to be linear when one takes scalings by natural logarithms.

First we recall from high school the logarithm laws, the index laws and the inverse relationship between logarithms and exponentials[7] (*You MUST memorise ALL of them!*).

Logarithm Laws

$$\ln(xy) = \ln x + \ln y$$
$$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$$
$$\ln(x^y) = y \ln x$$

[7]Obviously these laws hold for logarithms and exponentials of bases different to e but we will predominantly be using natural base logarithms and exponentials.

Exponential Laws

$$e^{x+y} = e^x e^y$$

$$e^{x-y} = \frac{e^x}{e^y}$$

$$e^{xy} = (e^x)^y$$

Inverse Relationship

$$\ln(e^x) = x = e^{\ln x}$$

Change of Base[8]

$$\log_b x = \frac{\ln x}{\ln b}$$

$$b^x = e^{x \ln b}$$

[8]Thus all logarithms and exponentials that are not base e can be converted.

Now we will practise our algebra with exponentials and logarithms:

Example
Solve for x: $e^{5x} = 3$.

Taking logarithms of both sides, we get $5x = \ln 3$. So that

$$x = \frac{\ln 3}{5} = 0.220.$$

Example
Solve for x: $\ln(3x + 2) = 5$.

Taking exponentials[9] of both sides, we get $3x + 2 = e^5$. So that

$$x = \frac{e^5 - 2}{3} = 48.8.$$

[9] Called "exponentiating".

Example

Solve for y: $\ln y = 3x + 5$.

Exponentiating of both sides, we get $e^{\ln y} = e^{3x+5} = e^5 e^{3x} \approx 148e^{3x}$.

$\therefore\ y = 148\,e^{3x}$

Example

Solve for y: $\ln y = 3\ln x + 7$.

Exponentiating of both sides, we get $e^{\ln y} = e^{3\ln x+7} = e^7 \left(e^{\ln x}\right)^3$. So that $y = 1096x^3$.

Solving Allometric (commonly known as "Power") Laws

Suppose that $y = ax^n$. Taking logarithms on both sides we get

$$\begin{aligned} \ln y &= \ln(ax^n) \\ &= \ln a + \ln x^n \\ &= \ln a + n \ln x \end{aligned}$$

Setting

$$\begin{aligned} Y &= \ln y \\ X &= \ln x \\ b &= \ln a \\ m &= n, \end{aligned}$$

we have

$$Y = mX + b.$$

So, transforming y to $Y = \ln y$ and x to $X = \ln x$, produces a linear relationship between X and Y.

Conversely a linear relationship, $Y = mX + b$, leads to a power relationship between x and y.

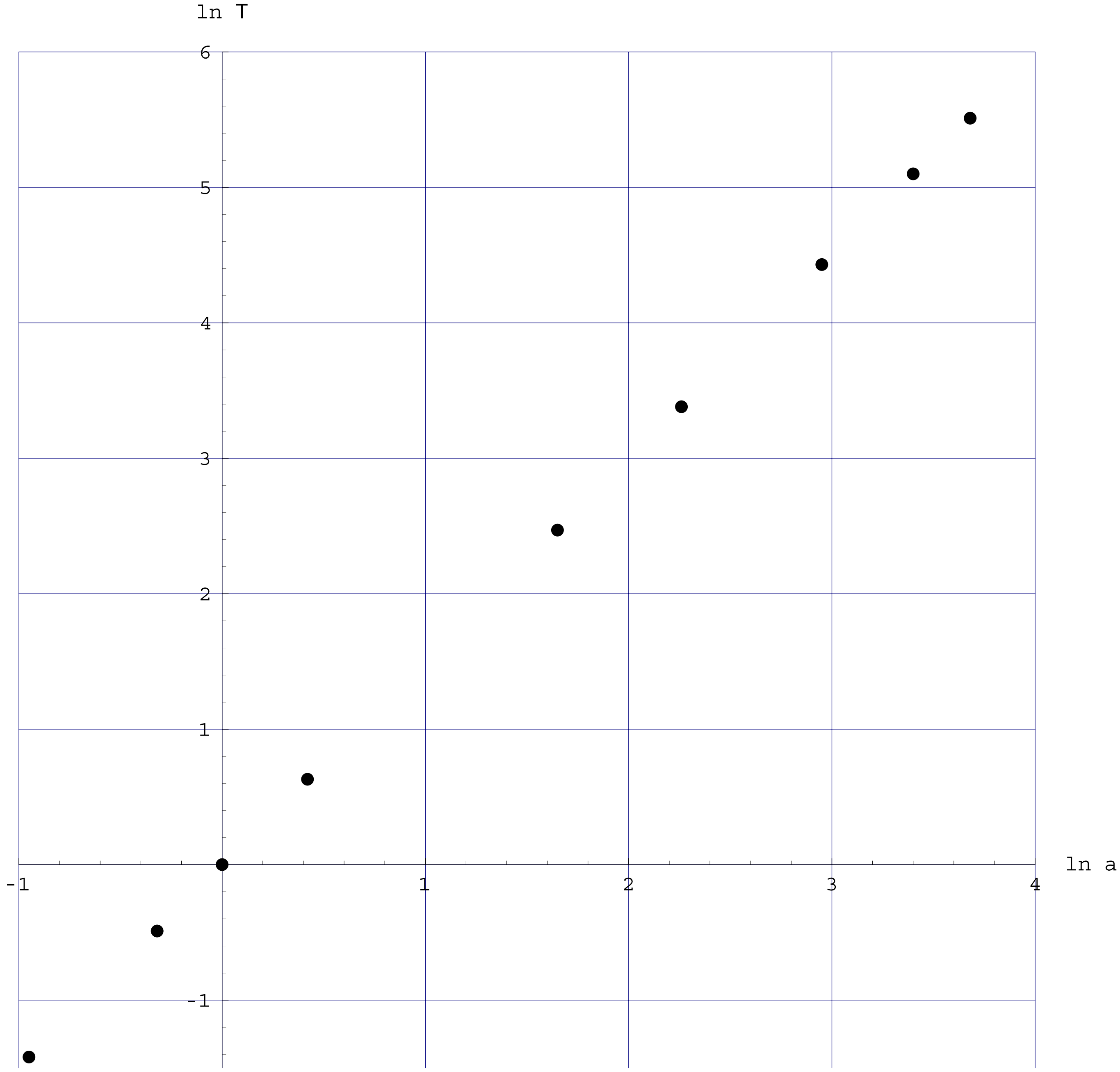
$$\begin{aligned} Y &= mX + b \\ \ln y &= m \ln x + b \\ e^{\ln y} &= e^{m \ln x + b} \\ y &= e^b \left(e^{\ln x}\right)^m \\ y &= ax^n. \end{aligned}$$

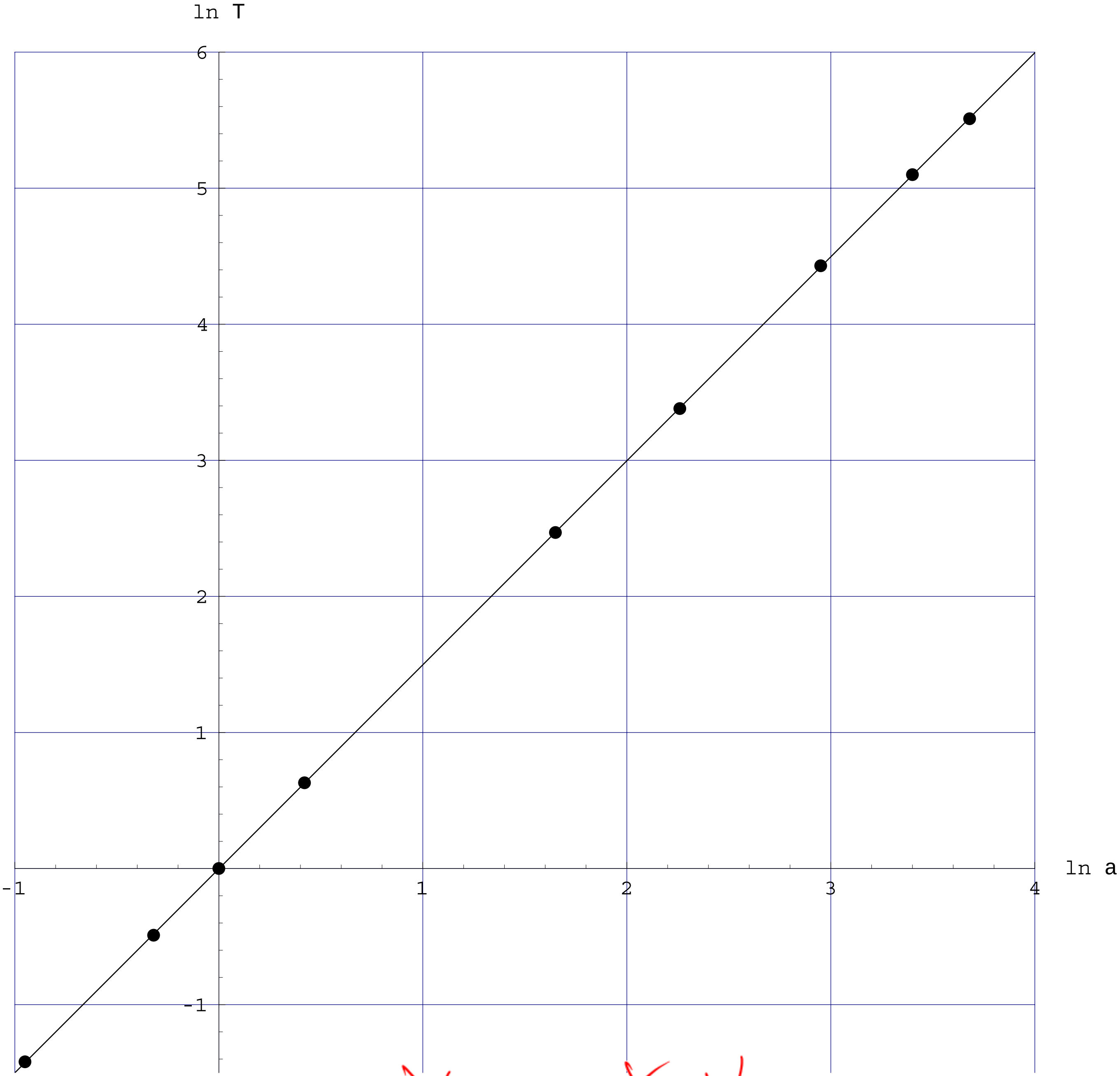
Allometric Laws

Let's take up the Kepler's third law example again.

Planet	M	V	E	M	J
a	0.387	0.723	1.000	1.524	5.204
$\ln a$	-0.95	-0.32	0.00	0.42	1.65
T	0.241	0.615	1.000	1.881	11.86
$\ln T$	-1.42	-0.49	0.00	0.63	2.47
Planet	S	U	N	P	
a	9.539	19.18	30.06	39.5	
$\ln a$	2.26	2.95	3.40	3.68	
T	29.46	84.0	164.8	248	
$\ln T$	3.38	4.43	5.10	5.51	

We plot $\ln T$ against $\ln a$.


ln T
6
5
4
3
2
1
-1
ln a
-1
1
2
3
4



$Y = mX + b$

We find that $\ln T = \frac{3}{2} \ln a$
ie $T = a^{\frac{3}{2}}$, showing us that Kepler's law is an allometric law.

Example

A researcher finds the following data.

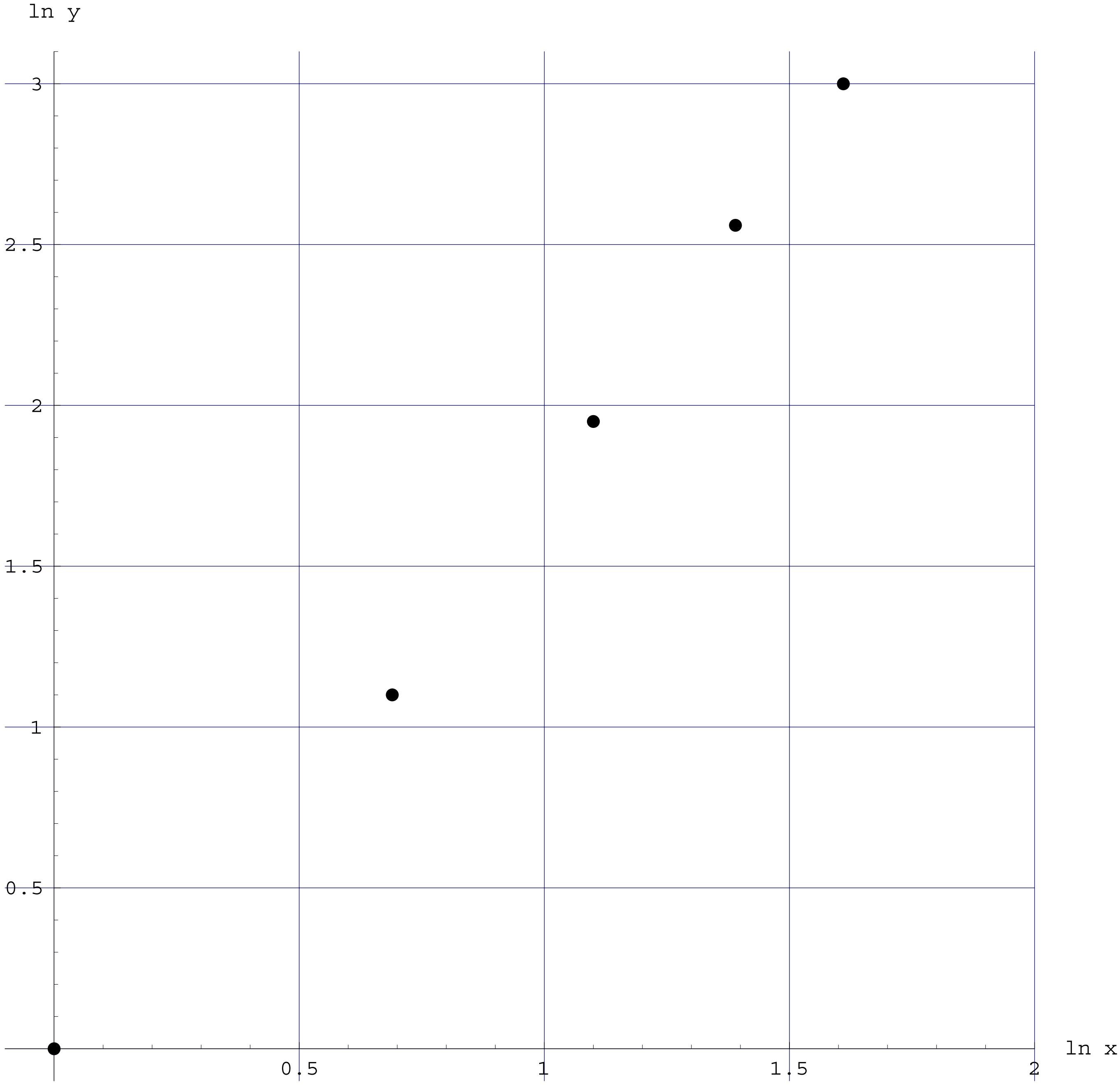
x	1	2	3	4	5
y	1	3	7	13	20

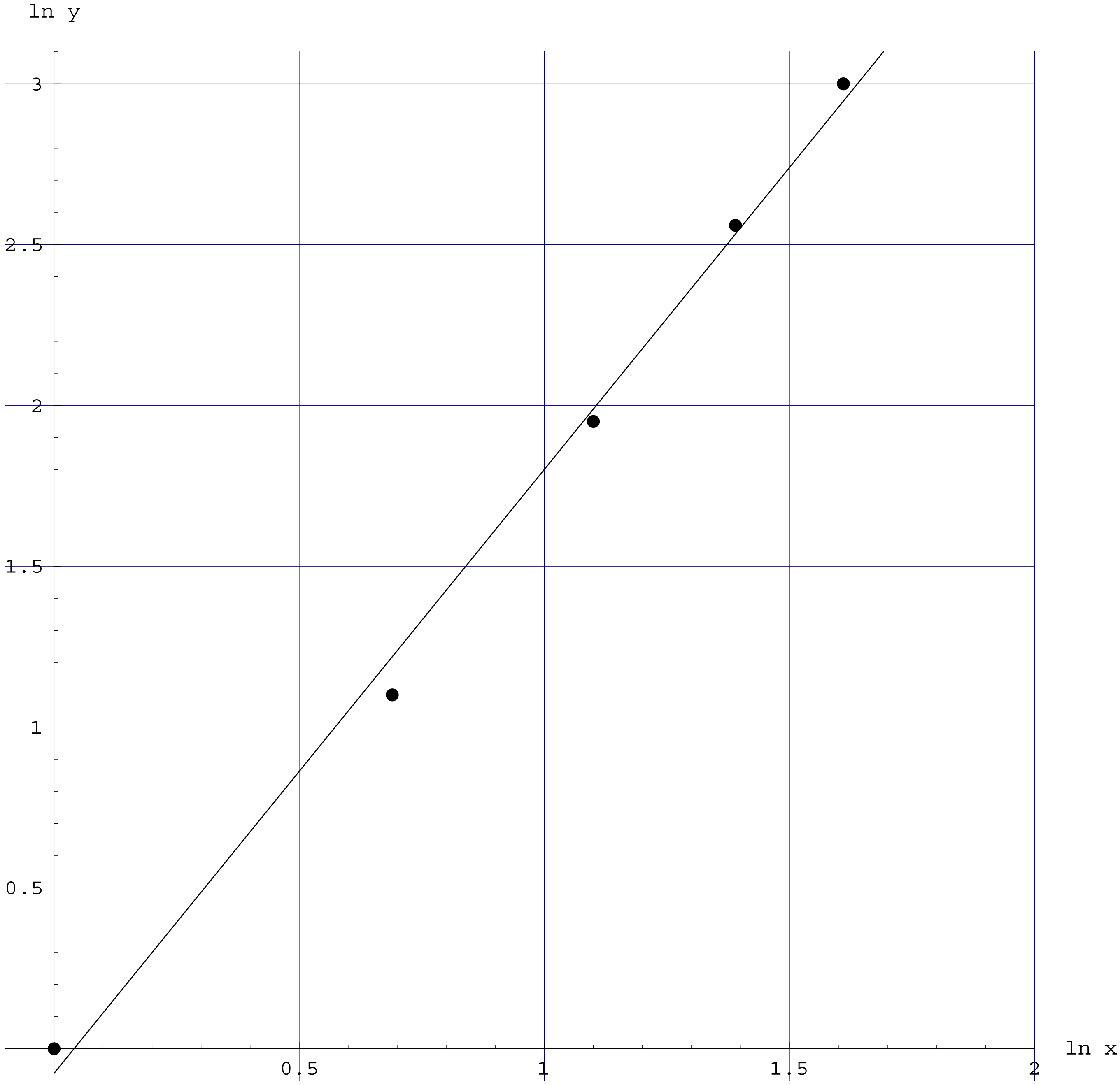
She suspects that the data conforms to a power law. Find a model which fits the data.

We tabulate $X = \ln x$ and $Y = \ln y$.

$X = \ln x$	0	0.69	1.10	1.39	1.61
$Y = \ln y$	0	1.10	1.95	2.56	3.00

We plot $\ln y$ against $\ln x$.


ln y
3
2.5
2
1.5
1
0.5
0.5
1
1.5
2
ln x


ln y
3
2.5
2
1.5
1
0.5
ln x
0.5
1
1.5
2

The line must be $Y = mX + b$, so we find[10]

$$\ln y = 1.88 \ln x - 0.08$$

Then

$$y = e^{-0.08} x^{1.88} = 0.92 x^{1.88}$$

[10]Notice how now here we have used accurate measurements of the actual line of best fit instead of using the first and last tabulated values. This gives a far more accurate results!

(cf MATH1013)

initial quantity

growth constant

Exponential Laws

Suppose that $y = Ae^{kx}$. Taking logarithms we get

$$\begin{aligned} \ln y &= \ln(Ae^{kx}) \\ &= \ln A + \ln e^{kx} \\ &= \ln A + kx \end{aligned}$$

Setting

$$\begin{aligned} Y &= \ln y \\ X &= x \\ b &= \ln A \\ m &= k, \end{aligned}$$

we have

$$Y = mX + b.$$

So, transforming y to $Y = \ln y$ (and trivially x to $X = x$), produces a linear relationship between X and Y. called a "semi-log" transformation

Conversely a linear relationship between $X = x$ and $Y = \ln y$
$Y = mX + b$, leads to an exponential relationship between x and y.

$$\begin{aligned} Y &= mX + b \\ \ln y &= mx + b \\ e^{\ln y} &= e^{mx+b} \\ y &= e^b \left(e^m\right)^x \\ y &= Ae^{kx}. \end{aligned}$$

Example

A researcher finds the following data.

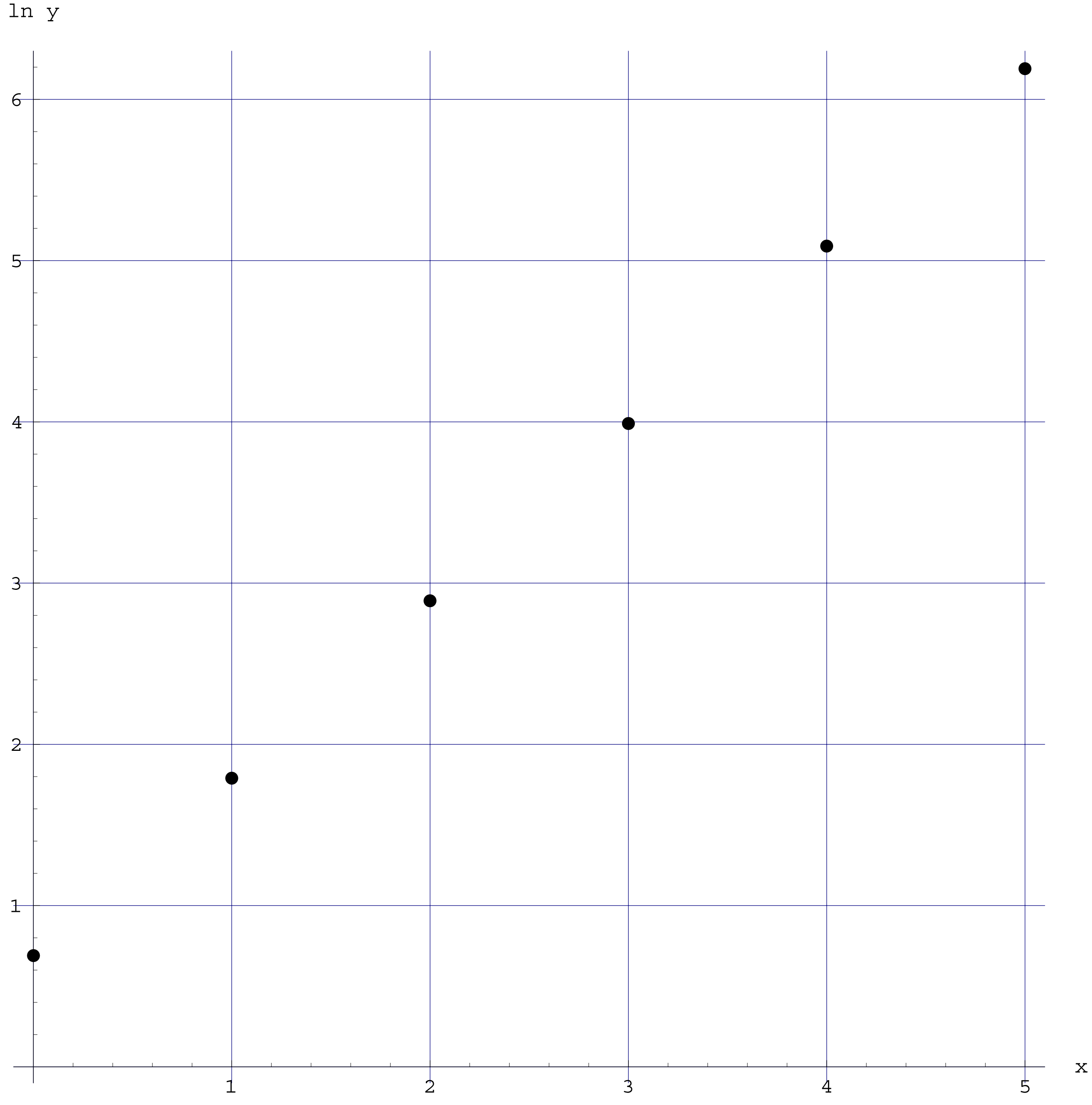
x	0	1	2	3	4	5
y	2	6	18	54	162	486

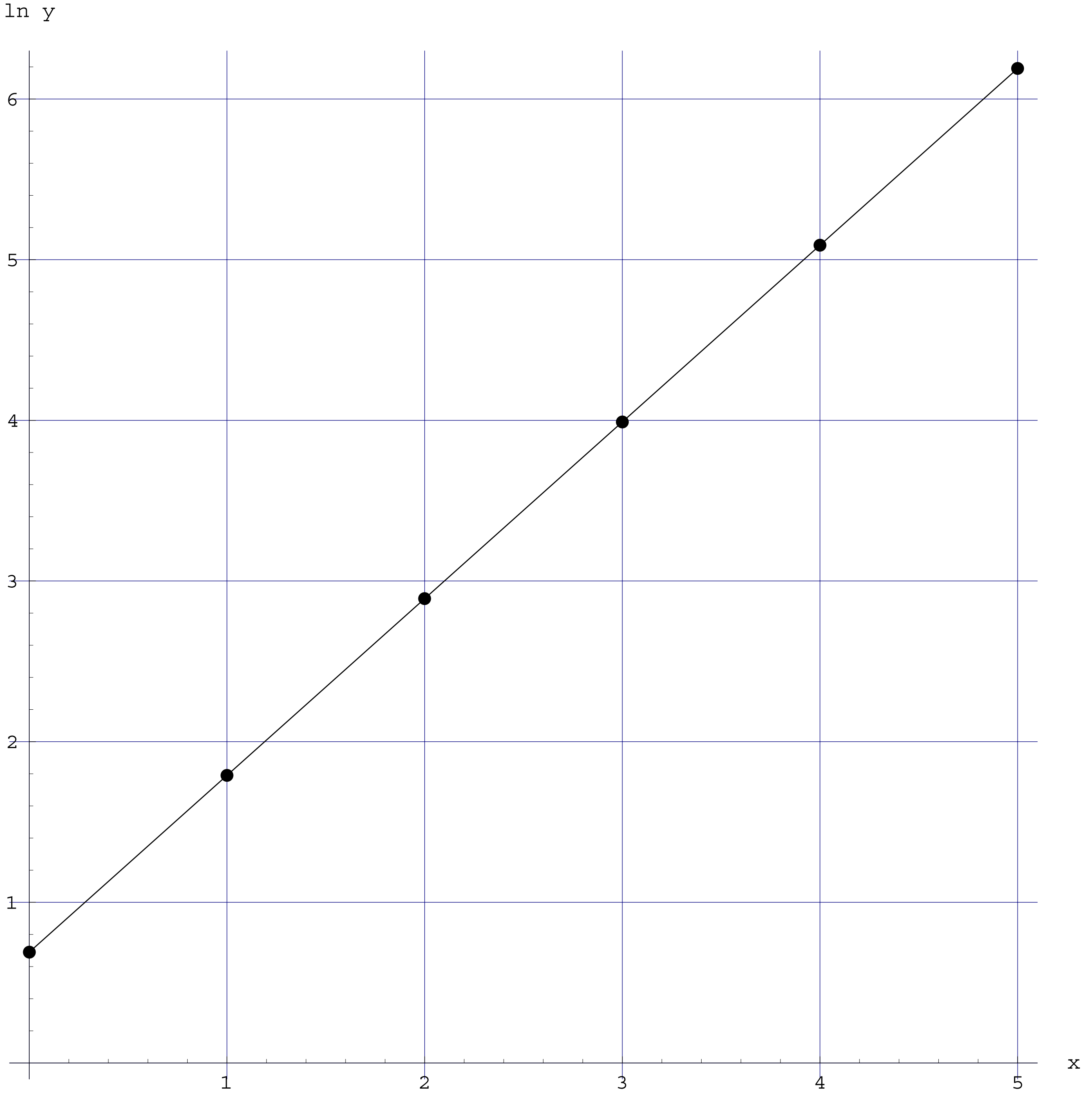
He suspects that the data conforms to an exponential law. Find a model which fits the data.

We tabulate $X = x$ and $Y = \ln y$.

x	0	1	2	3	4	5
$\ln y$	0.69	1.79	2.89	3.99	5.09	6.19

We plot $\ln y$ against x.


ln y
6
5
4
3
2
1
1
2
3
4
5
x



ln y
6
5
4
3
2
1
1
2
3
4
5
x

We find

$$Y = 1.10X + 0.69.$$

So that

$$\ln y = 1.10x + 0.69.$$

Then

$$\begin{aligned} y &= e^{\ln y} \\ &= e^{1.10x+0.69} \\ &= e^{0.69} \cdot e^{1.10x} \\ &\approx 1.99e^{1.10x}. \end{aligned}$$

Example

Use a semilog transformation to find a function to model the following data.

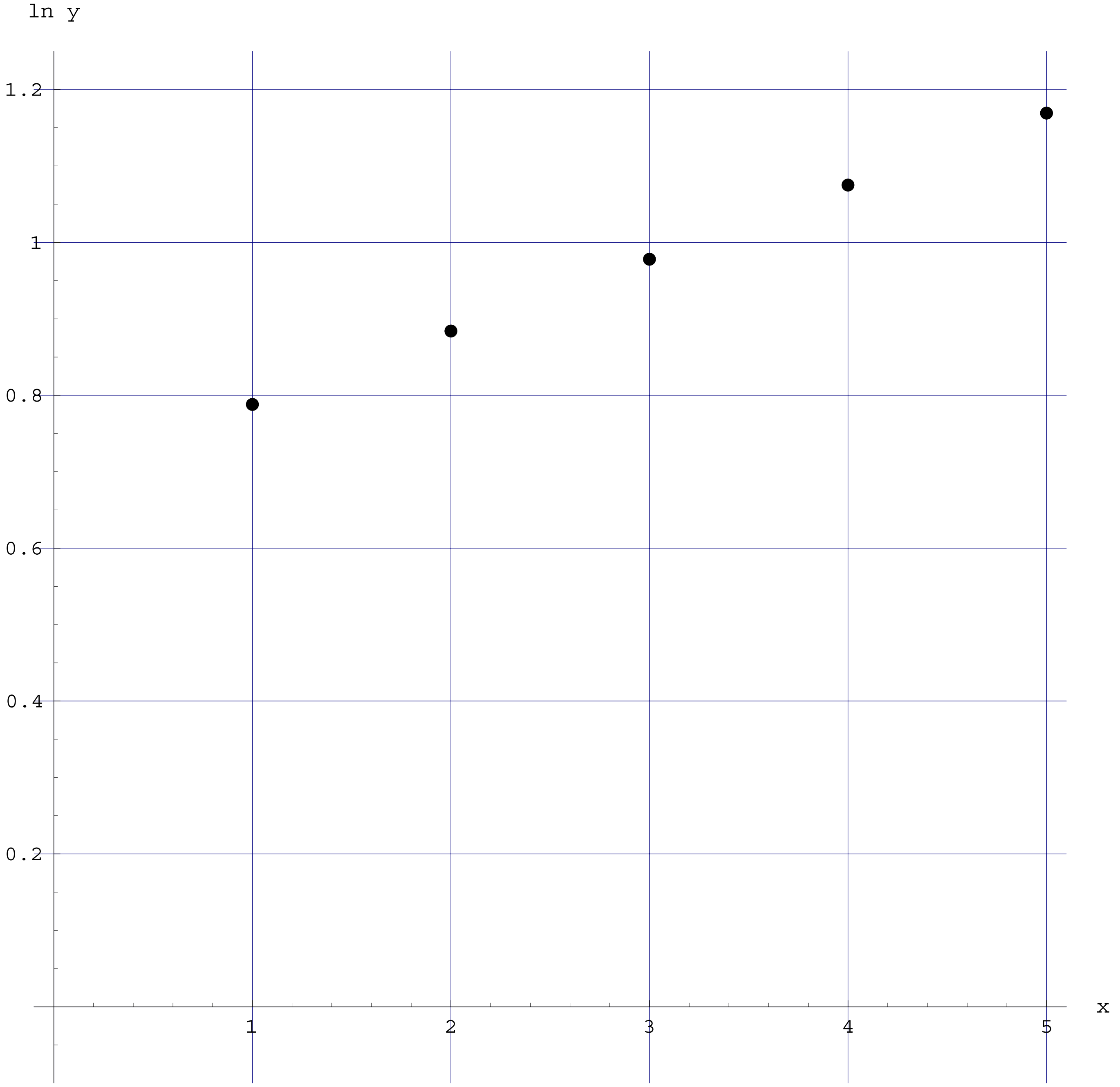
x	1	2	3	4	5
y	2.20	2.42	2.66	2.93	3.22

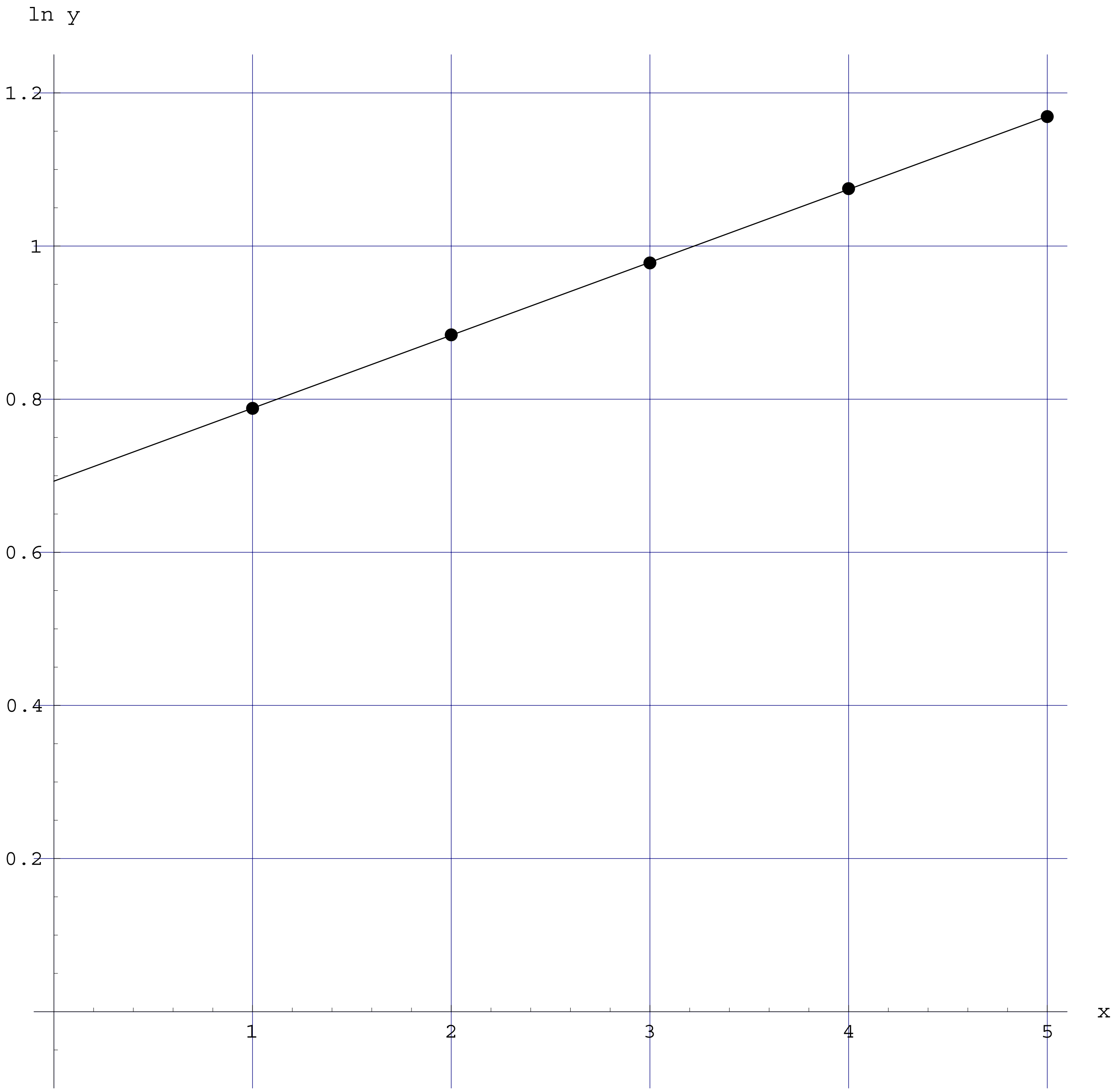
We tabulate $X = x$ and $Y = \ln y$.

X	1	2	3	4	5
Y	0.788	0.884	0.978	1.075	1.169

We plot[11] $\ln y$ against x.

[11]The following graph is an example of poorly scaled axes, and is a common error in data plotting among students. When plotting data, make sure the ranges of your axes reflect the ranges of your data such that the entire plotting space is used and the data/line is spread out amongst the plot area, not bunched up in one corner like the following graph!


ln y
1.2
1
0.8
0.6
0.4
0.2
1
2
3
4
5
x


ln y
1.2
1
0.8
0.6
0.4
0.2
1
2
3
4
5
x

We want $Y = mX + b$, so we find

$$Y = 0.0953X + 0.693.$$

So that

$$\ln y = 0.0953x + 0.693.$$

Then

$$\begin{aligned} y &= e^{\ln y} \\ &= e^{0.0953x+0.693} \\ &= e^{0.693} \cdot e^{0.0953x} \\ &\approx 2e^{0.1x}. \end{aligned}$$

Example

A researcher finds the following data.

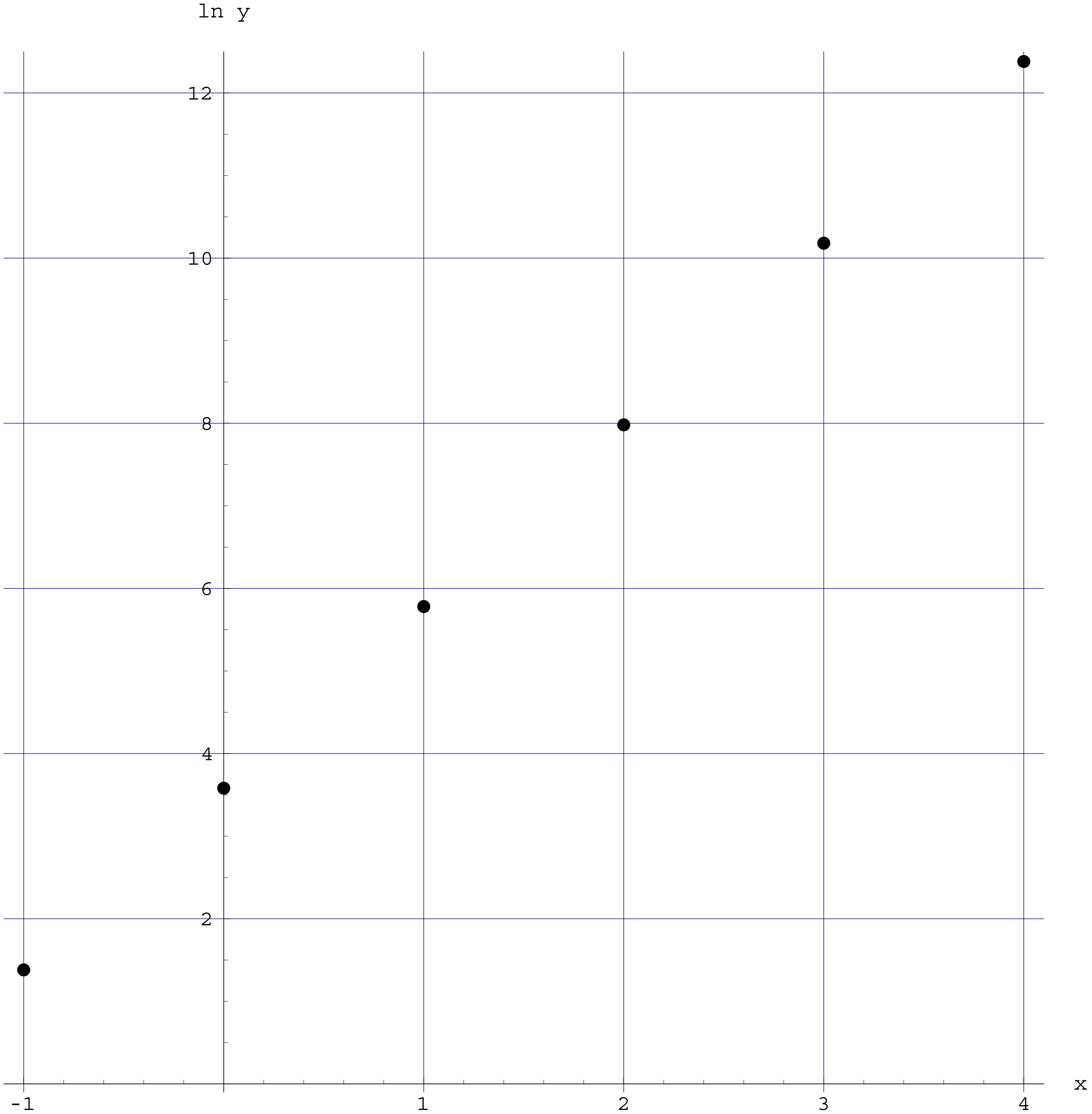
x	-1	0	1	2	3	4
y	4	36	324	2922	26370	237994

He suspects that the data conforms to an exponential law. Find a model which fits the data.

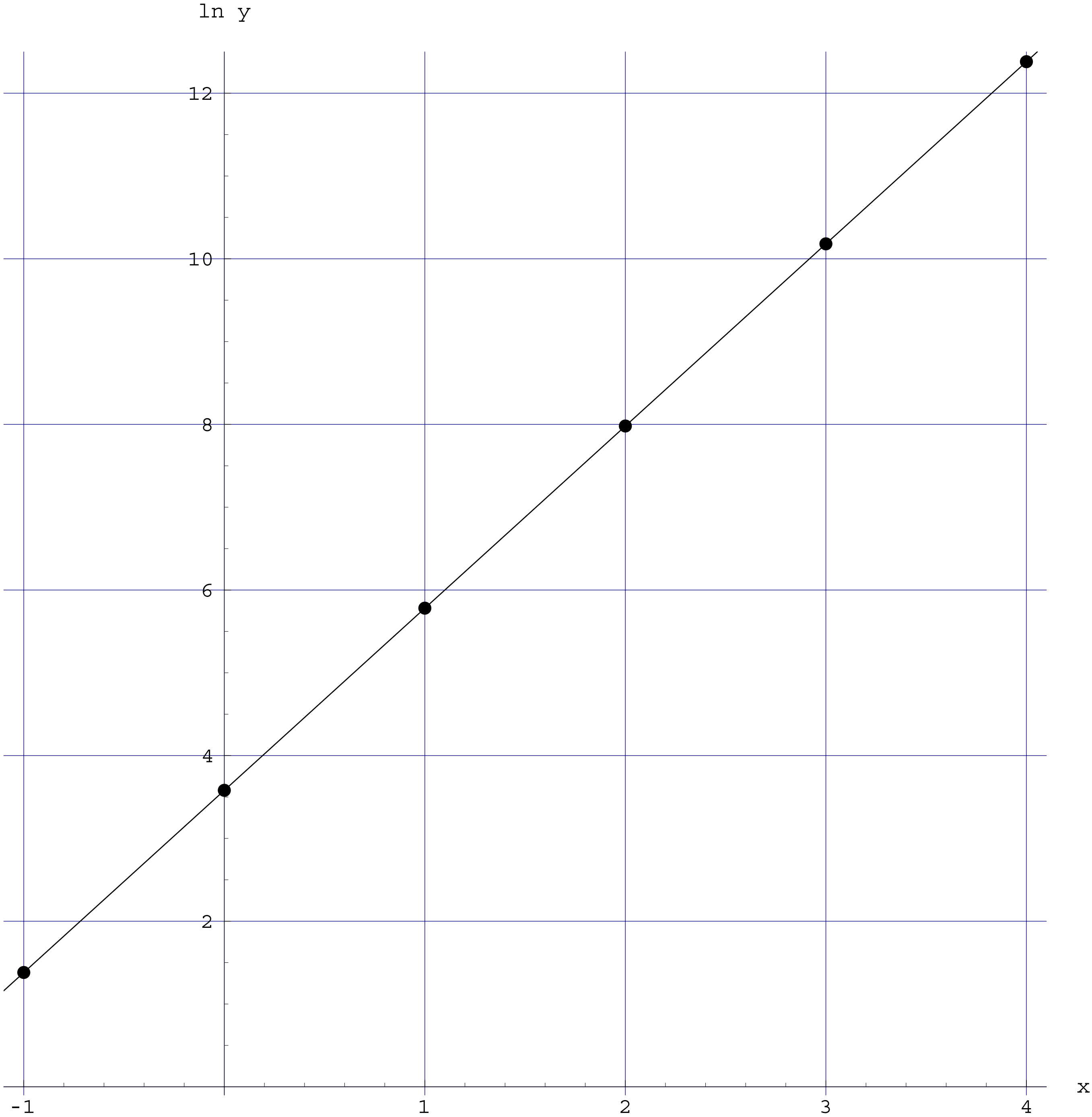
We tabulate $X = x$ and $Y = \ln y$.

x	-1	0	1	2	3	4
$\ln y$	1.38	3.58	5.78	7.98	10.18	12.38

We plot $\ln y$ against x.



ln y
12
10
8
6
4
2
-1
1
2
3
4
x


ln y
12
10
8
6
4
2
-1
1
2
3
4
x

We find

$$Y = 2.20X + 3.58$$

So that

$$\ln y = 2.20x + 3.58.$$

Then

$$\begin{aligned} y &= e^{\ln y} \\ &= e^{2.20x+3.58} \\ &= e^{3.58}\left(e^{2.20}\right)^{x} \\ &\approx 35.9e^{2.20x}. \end{aligned}$$

Example

Use a semi-log[12] transformation to find a function to model the following data.

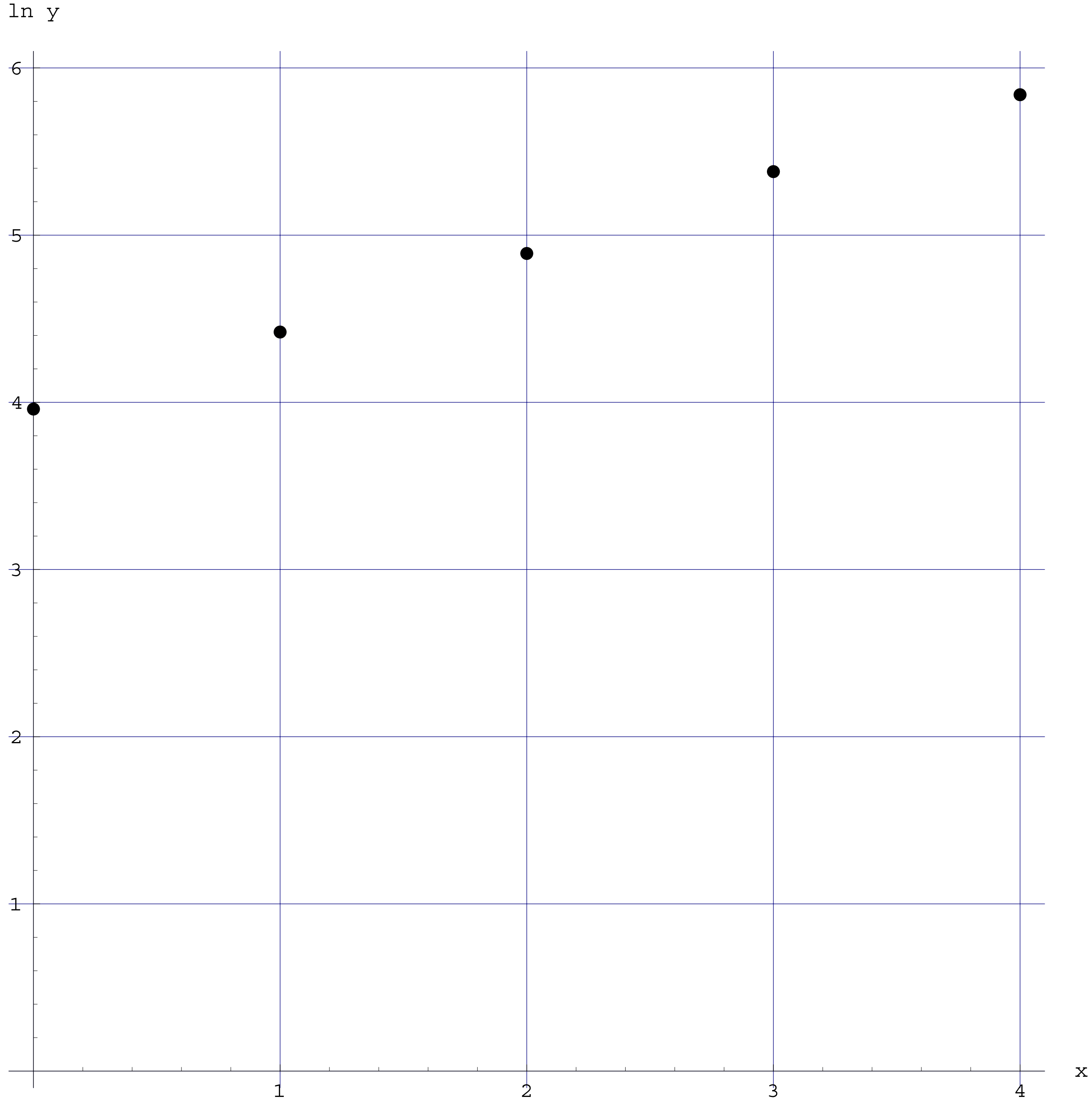
x	0	1	2	3	4
y	52.5	83.1	133.0	217.0	343.8

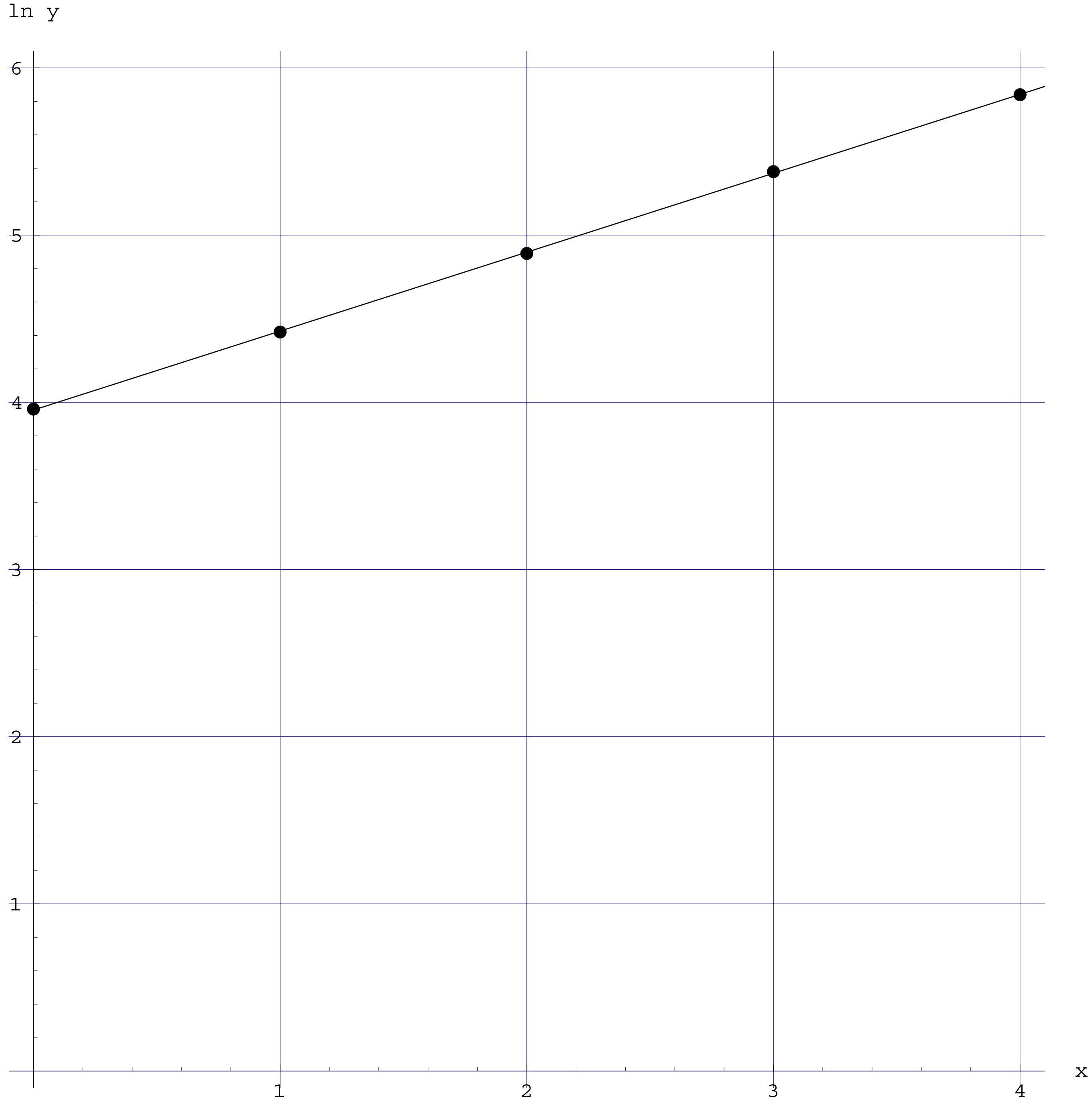
We tabulate $X = x$ and $Y = \ln y$.

x	0	1	2	3	4
$\ln y$	3.96	4.42	4.89	5.38	5.84

We plot $\ln y$ against x.

[12] "Semi-log" means to take logs of only one variable. You may wonder how we determine which variable to transform. Notice how x increases regularly, but the y values increase irregularly. We want to transform away the irregular increase, thus take logarithms of the y values, NOT the x values.


ln y
6
5
4
3
2
1
1
2
3
4
x


ln y
6
5
4
3
2
1
1
2
3
4
x

We find

$$Y = 0.472X + 3.954.$$

So that

$$\ln y = 0.472X + 3.954.$$

Then

$$\begin{aligned} y &= e^{\ln y} \\ &= e^{0.472X+3.954} \\ &= e^{3.954}\left(e^{0.472}\right)^{x} \\ &\approx 52.1e^{0.472x}. \end{aligned}$$

Finally, back to a power law example.

Example

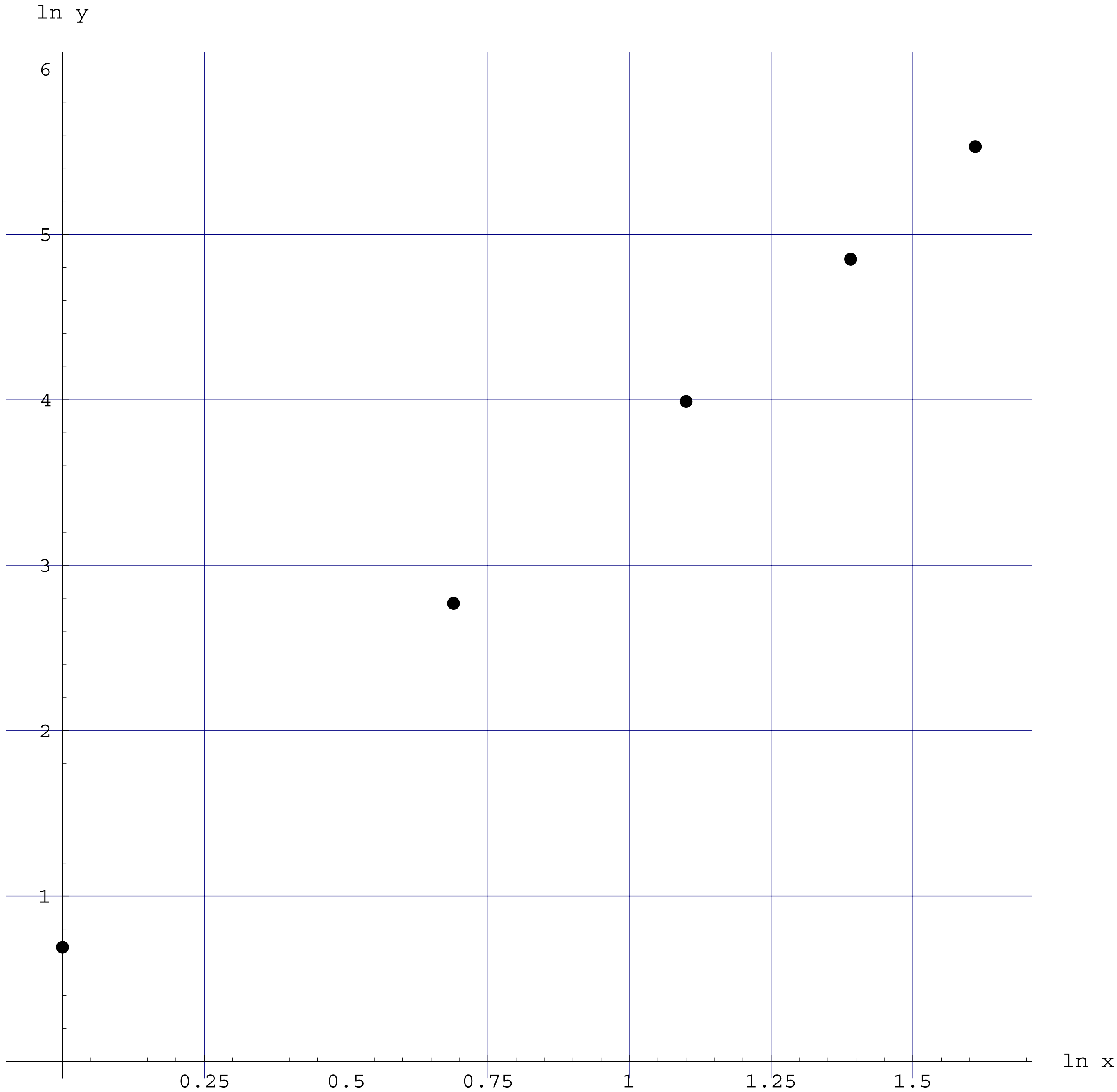
Use a log-log transformation to find a function to model the following data.

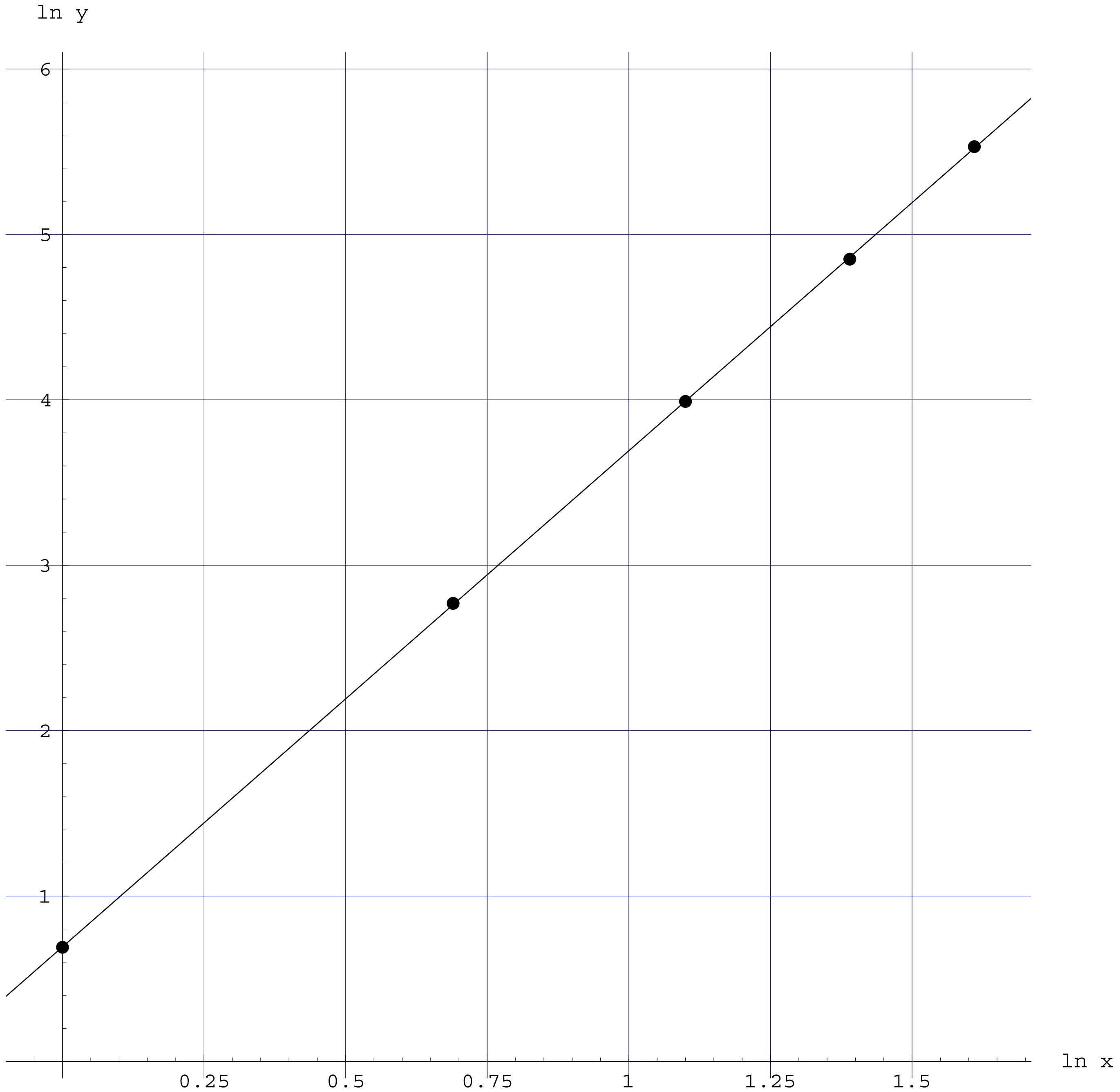
x	1	2	3	4	5
y	2	16	54	128	250

We tabulate $X = \ln x$ and $Y = \ln y$.

$X = \ln x$	0.00	0.69	1.10	1.39	1.61
$Y = \ln y$	0.69	2.77	3.99	4.85	5.52

We plot $Y = \ln y$ against $X = \ln x$.


ln y
6
5
4
3
2
1
ln x
0.25
0.5
0.75
1
1.25
1.5


ln y
6
5
4
3
2
1
ln x
0.25
0.5
0.75
1
1.25
1.5

We find

$$Y = 2.999X + 0.693.$$

So that

$$\ln y = 2.999 \ln x + 0.693.$$

Then

$$\begin{aligned} y &= e^{\ln y} \\ &= e^{2.999 \ln x + 0.693} \\ &= e^{0.693} \left(e^{\ln x}\right)^{2.999} \\ &\approx 2x^3. \end{aligned}$$

Example

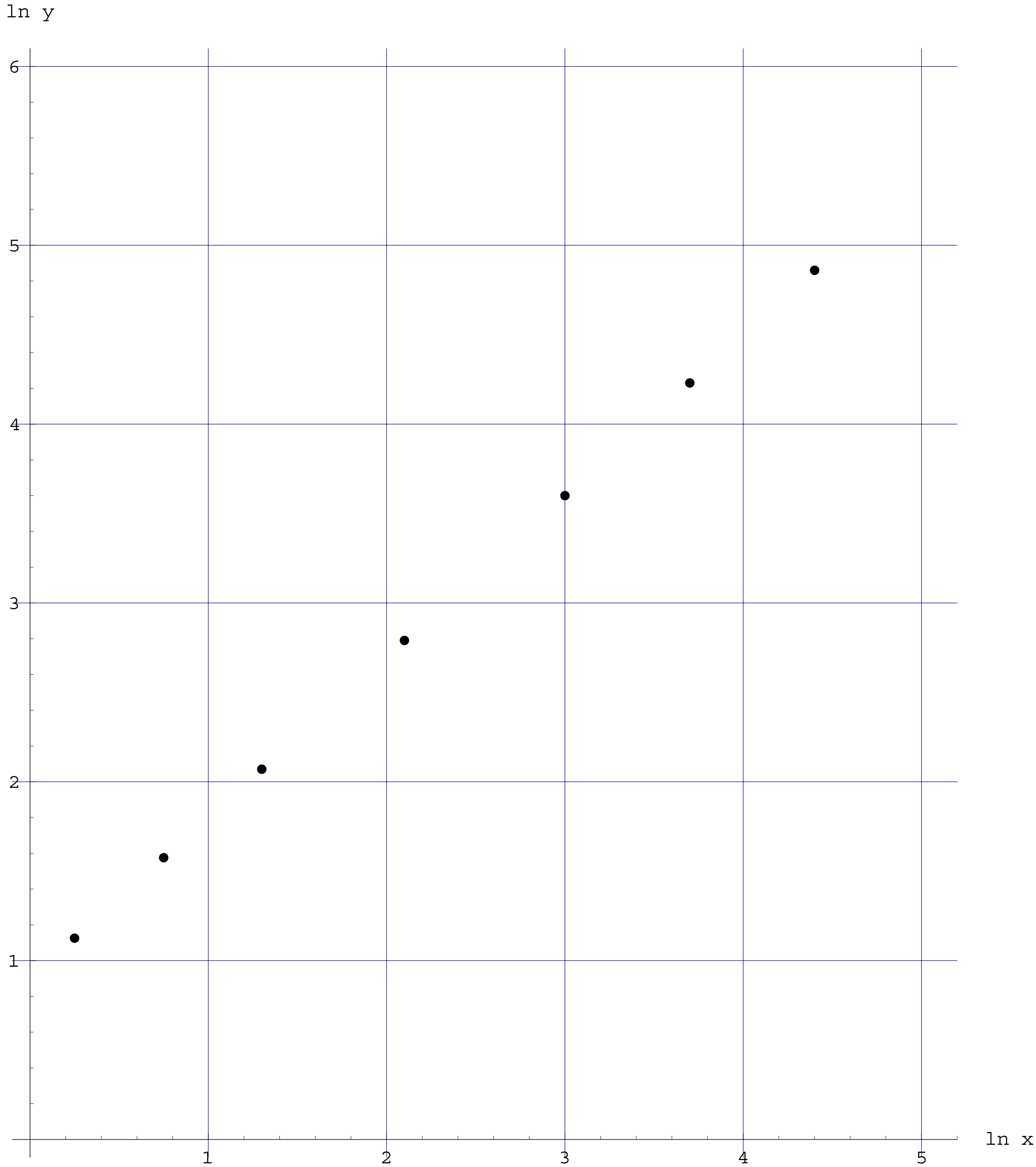
A researcher suspects that the following data can be explained by an allometric relationship between x and y. Show that this is so and find the relationship.

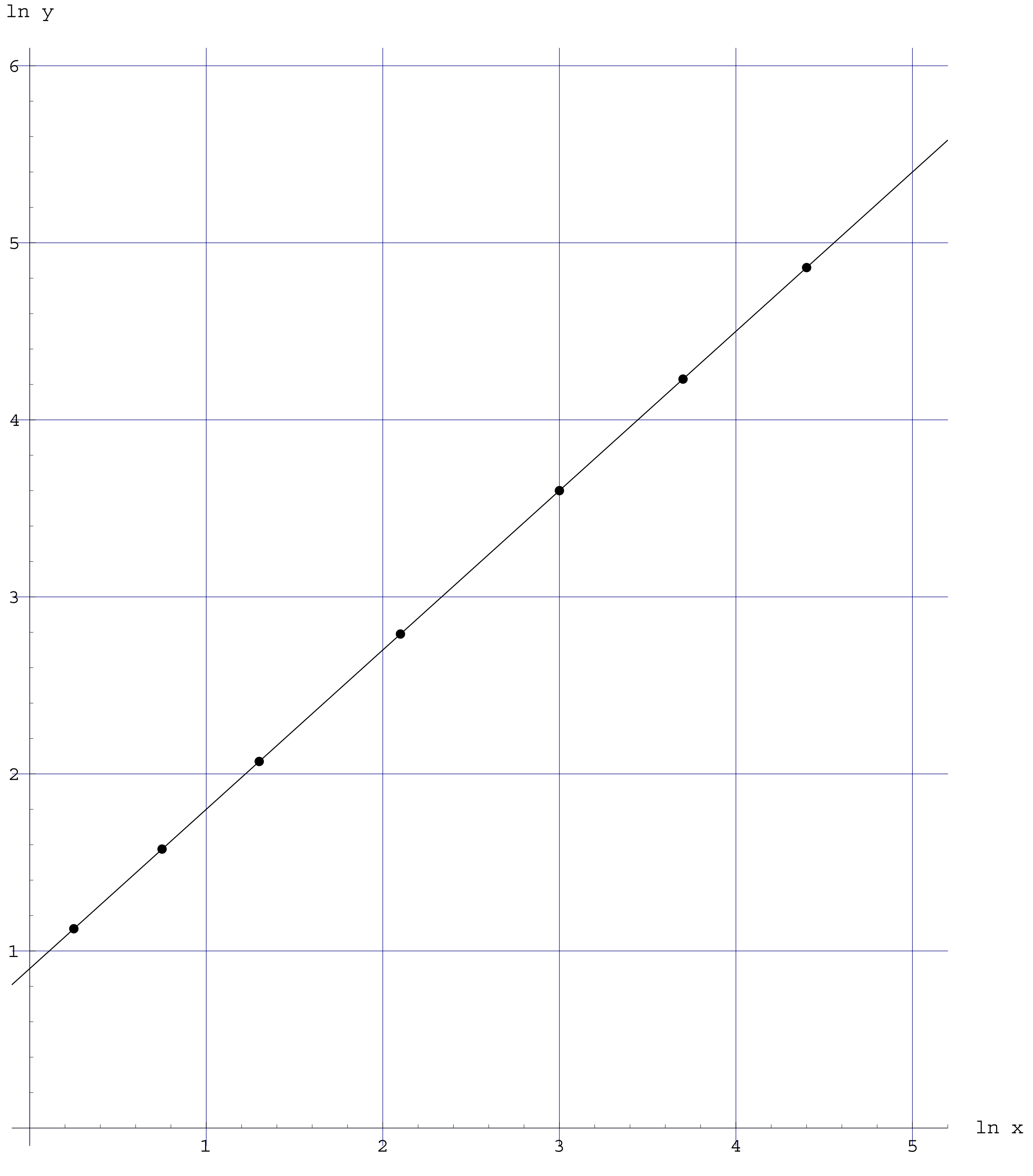
x	1.284	2.117	3.669	8.166
y	3.080	4.831	7.924	16.281
x	20.08	40.45	81.45	
y	36.598	68.717	129.024	

We tabulate $X = \ln x$ and $Y = \ln y$.

$X = \ln x$	0.25	0.75	1.30	2.10
$Y = \ln y$	1.125	1.575	2.07	2.97
$X = \ln x$	3.00	3.70	4.40	
$Y = \ln y$	3.60	4.23	4.86	

We plot $Y = \ln y$ against $X = \ln x$.


ln y
6
5
4
3
2
1
1
2
3
4
5
ln x


ln y
6
5
4
3
2
1
1
2
3
4
5
ln x

We find

$$Y = 0.9 + 0.9X.$$

So that

$$\ln y = 0.9 \ln x + 0.9.$$

Then

$$\begin{aligned} y &= e^{\ln y} \\ &= e^{0.9 \ln x + 0.9} \\ &= e^{0.9} \left(e^{\ln x}\right)^{0.9} \\ &\approx 2.5x^{0.9} \end{aligned}$$

Example

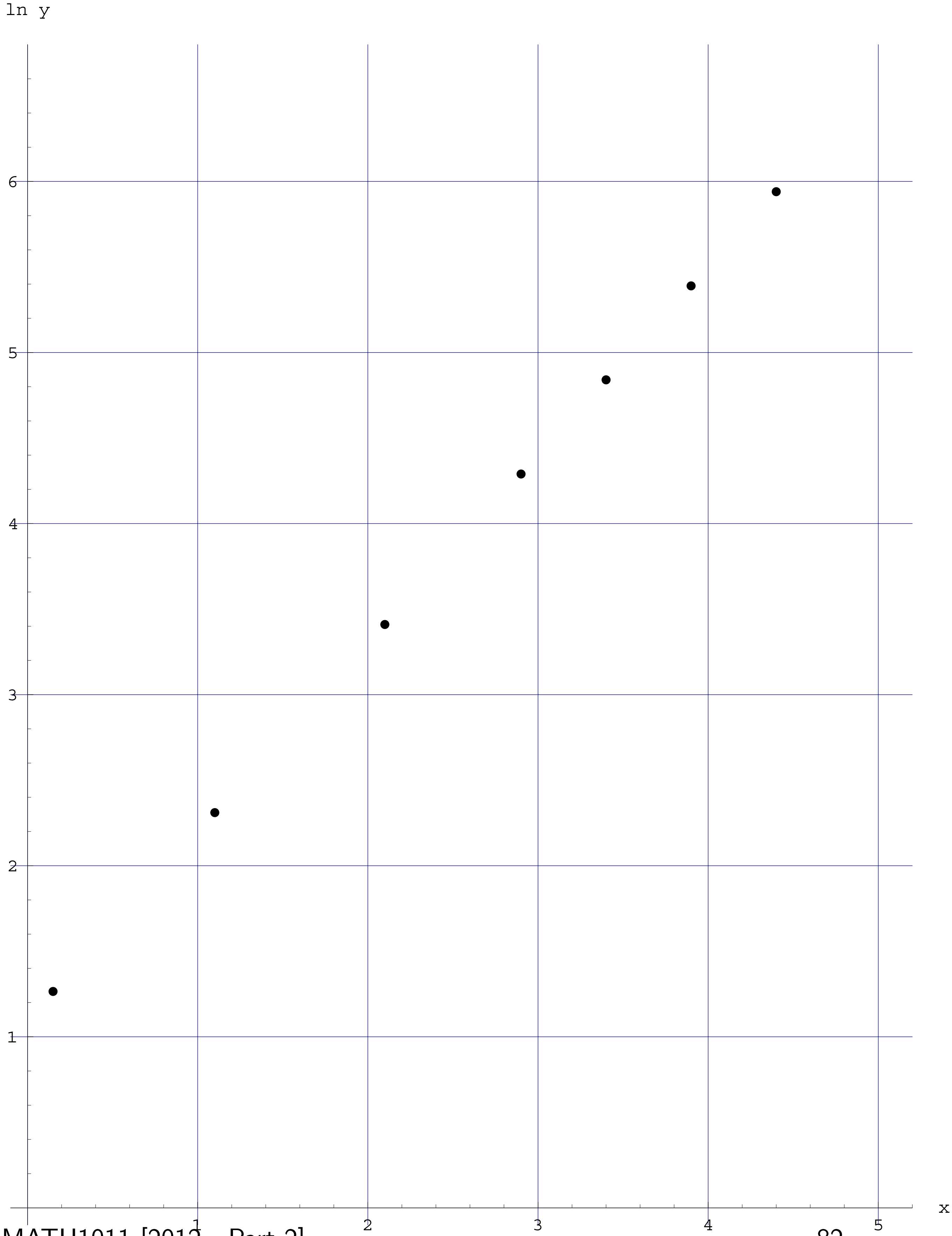
A researcher suspects that the following data can be explained by an exponential relationship between x and y. Show that this is so and find the relationship.

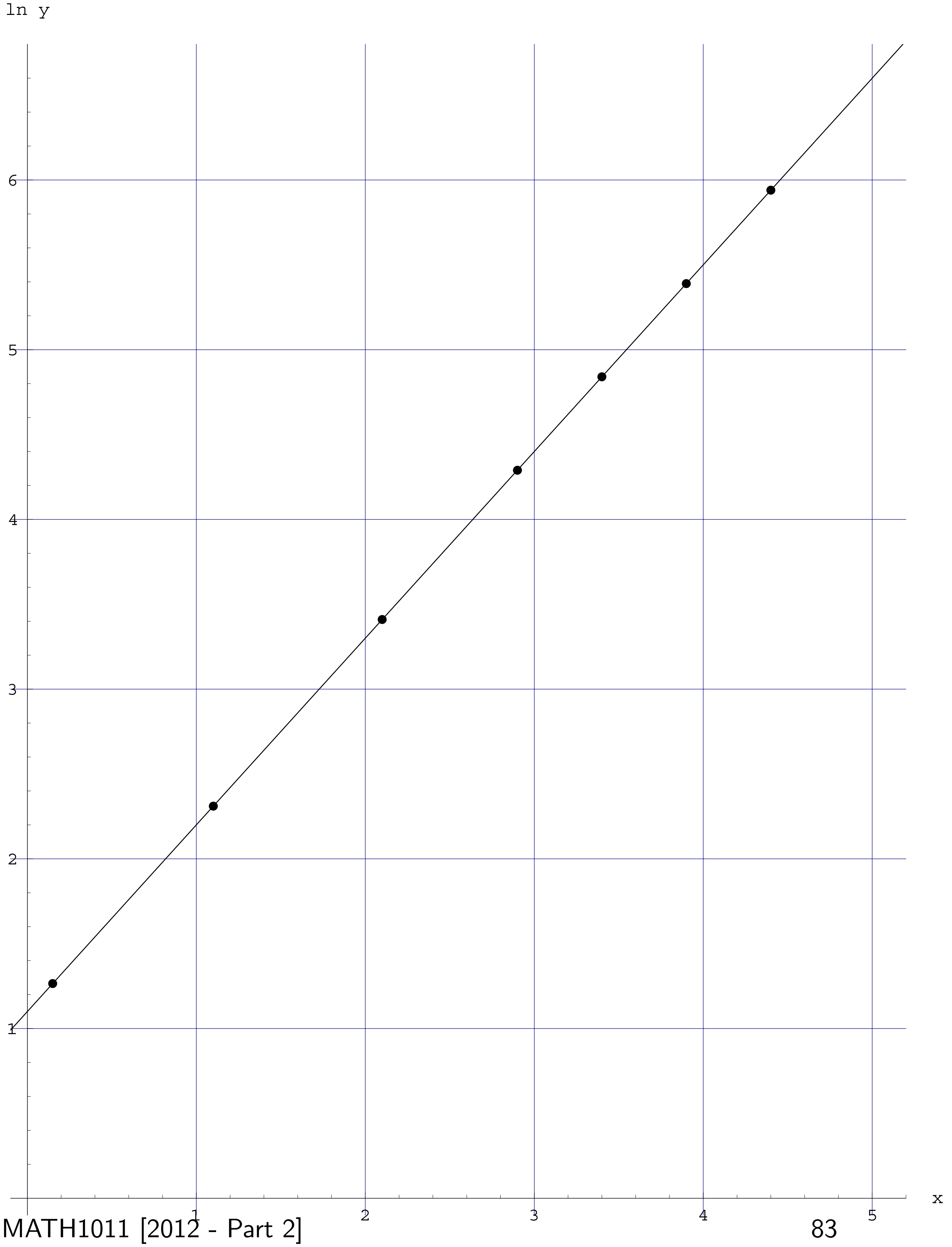
x	0.15	1.10	2.10	2.90
y	3.54	10.07	30.27	72.97
x	3.40	3.90	4.40	
y	126.47	219.20	379.93	

We tabulate $X = x$ and $Y = \ln y$.

$X = x$	0.15	1.10	2.10	2.90
$Y = \ln y$	1.27	2.31	3.41	4.29
$X = x$	3.40	3.90	4.40	
$Y = \ln y$	4.84	5.39	5.94	

We plot $Y = \ln y$ against $X = x$.


ln y
6
5
4
3
2
1
1
2
3
4
5
x


ln y
6
5
4
3
2
1
1
2
3
4
5
x

We find

$$Y = 1.1 + 1.1X.$$

So that

$$\ln y = 1.1x + 1.1.$$

Then

$$\begin{aligned} y &= e^{\ln y} \\ &= e^{1.1x+1.1} \\ &= e^{1.1}e^{1.1x} \\ &\approx 3.0e^{1.1x} \end{aligned}$$

Logarithmic Scales of Measurement

[*§1.3.4 of Notes*]

The fact that logarithmic transformations of data often make experimental data follow a linear model suggests that, maybe, some quantities should be measured by using a logarithmic scale. This is, in fact, often done: examples are as follows:

- Intensity level of sound—decibels
- Richter scale for earthquakes
- pH level—acidity
- Musical pitch

Human senses do, in fact, behave in a “logarithmic way.”

Since there are logarithms to base 10 in the first three examples and logarithms to base 2 in the last, we need a short review of logarithms to any base $a > 0$.

$\log_b x$ is defined by

$$\boxed{x = b^a \Leftrightarrow a = \log_b x.}$$

In other words,

“*the logarithm of a number is the index of its base*”.

This entails $\log_b b^x = x = b^{\log_b x}$ (the inverse relation) and we can derive the generalised[13] change of base law:

$$\boxed{\log_b x = \frac{\log_a x}{\log_a b}.}$$

[13] Simply set $a = e$ to get the previous law we saw earlier.

Sound

The intensity of a sound wave is the power transfered through a unit area perpendicular to the direction of propagation. I measured in Watts per square meter $(\mathrm{W} \cdot \mathrm{m}^{-2})$ is proportional to the square of the frequency times the square of the amplitude of the sound wave.

The faintest audible sound is $I_0 = 10^{-12}\ \mathrm{W} \cdot \mathrm{m}^{-2}$.

The pain threshold is $1\ \mathrm{W} \cdot \mathrm{m}^{-2}$.

The pain threshold is 10^{12} times the faintest audible sound. So sound intensity is measured by a logarithmic scale of base 10.

A sound of intensity I is said to have loudness[14]

$$L(I) = 10 \log_{10} \left(\frac{I}{I_0} \right) \text{ decibels.}$$

Then the faintest audible sound has loudness $10 \log_{10} 1 = 0$ decibels. The pain threshold has loudness $10 \log_{10} 10^{12} = 120$ decibels.

[14]The ratio I/I_0 is the *relative* intensity and ratios are very commonly used!

To human senses a 70 decibel sound seems ten times louder than a 60 decibel sound, whilst as a 70 decibel sound seems a hundred times louder than a 50 decibels sound.

So perceived sound loudness increases in apparent equal steps, but the actual intensity increases exponentially.

Music

Let's turn to a base 2 application:

The pitch of a musical note depends on the frequency of the sound wave but a logarithmic scale is used.

Here are the frequencies of "A" notes in Hz. The interval between successive A notes is one octave: the frequency doubles.

27.5	55	110	220	440	880	1760	3520

One octave has 12 semitones. So increase in pitch in semitones is

$$12 \log_2 \left(\frac{\text{frequency of high note}}{\text{frequency of low note}} \right)$$

To human senses the octave jumps seem to be equal steps in increase in pitch but the frequency doubles at each step.

Earthquakes

As for sound waves the intensity of an earthquake is the power per unit area. Let I_0 be the intensity of background noise (ie when there is no earthquake). The Richter magnitude is given by

$$\log_{10}\left(\frac{I}{I_0}\right)$$

So an earthquake of magnitude 7 has ten times the power passing through a unit area as an earthquake of magnitude 6 and a hundred times as much as an earthquake of magnitude 5.

Chemistry

The acidity/alkalinity of a solution is determined by the concentration of Hydrogen ions $[\mathrm{H}^+]$ measured in $\mathrm{mol} \cdot \mathrm{L}^{-1}$.

	$[\mathrm{H}^+]$
Acid	10^{-2} to 10^{-7}
Neutral	10^{-7}
Alkaline	10^{-7} to 10^{-12}

$$\mathrm{pH} = -\log_{10}[\mathrm{H}^+]$$

$\Leftrightarrow [H^+] = 10^{-pH}$

	pH
Acid	2 to 7
Neutral	7
Alkaline	7 to 12

So a change of $+1$ in pH corresponds to $[\mathrm{H}^+] \div 10$.

Example

Two solutions S_1 and S_2 have pH factors of 4.7 and 7.5 respectively. Which solution is more acidic and by how many times?

$pH_{S_2} - pH_{S_1} = 2.8$

S_1 is the more acidic.

For S_1 $\qquad [\mathrm{H}^+] = 10^{-4.7}$.

For S_2 $\qquad [\mathrm{H}^+] = 10^{-7.5}$.

Ratio of $[\mathrm{H}^+]$ of the two solutions is

$$\frac{10^{-4.7}}{10^{-7.5}} = 10^{-4.5-(-7.5)} = 10^{2.8} = 631$$