Series

[*§3.1 of Notes, §12.2 & Appendix E of Stewart*]

$$\sum_{n=K}^{N} f(n)$$

"For each number n from K to N you write down $f(n)$ (by substituting n into f) and then add them up"

Examples

$$\sum_{k=1}^{7} k = 1+2+3+4+5+6+7 = 28$$

$$\sum_{\ell=1}^{4} \ell^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30$$

$$\sum_{n=1}^{6} 3 = 3+3+3+3+3+3 = 6\times 3 = 18$$

$$\sum_{r=1}^{20} \frac{r}{r+2} = \tfrac{1}{3} + \tfrac{2}{4} + \tfrac{3}{5} + \cdots + \tfrac{20}{22}$$

$$\sum_{k=0}^{4}(3k^2+5k-4)$$

$$= (3\times 0^2+5\times 0-4)+(3\times 1^2+5\times 1-4)$$

$$+ (3\times 2^2+5\times 2-4)+(3\times 3^2+5\times 3-4)$$

$$+ (3\times 4^2+5\times 4-4)$$

$$= -4+4+18+38+64=120$$

So we can split the above sum up separately:

$$\sum_{k=0}^{4}(3k^2+5k-4)$$

$$= 3\sum_{k=0}^{4}k^2+5\sum_{k=0}^{4}k-4\sum_{k=0}^{4}1$$

$$= 3(0^2+1^2+2^2+3^2+4^2)$$

$$+ 5(0+1+2+3+4)-4(1+1+1+1+1)$$

$$= 3\times 30+5\times 10-4\times 5$$

$$= 90+50-20=120$$

In that previous example we used the *linearity* of summation to split up the sum. The general rule is

$$\boxed{\sum_k [af(k) + bg(k)] = a\sum_k f(k) + b\sum_k g(k)}$$

So now we know how to expand sums, what about taking a series and writing it in sigma notation?

Example

Write $\frac{1}{1^2} + \frac{3}{2^2} + \frac{5}{3^2} + \cdots + \frac{19}{10^2}$ using sigma notation:

$$\sum_{n=1}^{10} \frac{2n-1}{n^2}$$

or

$$\sum_{k=0}^{9} \frac{2n+1}{(n+1)^2}$$

Shifting indices

The two previous sums are identical even though they look different. In effect, we have just shifted the indices. To sum up 10 elements, you can either go from index 0 to 9, or from index 1 to 10. If the start/finish indices are shifted, then your f formula also needs to be shifted. In particular:

$$\boxed{\sum_{n=K}^{N} f(n) = \sum_{n=K+1}^{N+1} f(n-1)}$$

Some more examples:[1]

$$\sum_{k=0}^{n} \frac{1}{k!} = 1 + 1 + \tfrac{1}{2} + \tfrac{1}{6} + \cdots + \tfrac{1}{n!} \text{ and}$$

$$\sum_{s=0}^{n} \frac{1}{5^s} = 1 + \tfrac{1}{5} + \tfrac{1}{25} + \cdots + \tfrac{1}{5^n}$$

Some sums are easy to calculate:

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \tfrac{1}{2} n(n+1)$$

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[1] $k! := k(k-1)(k-2)\ldots 3 \times 2 \times 1$

Here's why:[2]

$$2\sum_{k=1}^{n} k$$

$$= 1 + 2 + \cdots + n$$

$$\quad n + (n-1) + \cdots + 1$$

$$= n(n+1)$$

So

$$\sum_{k=1}^{100} k = \tfrac{1}{2} \times 100 \times 101 = 5050$$

We can use this result to sum any arithmetic progression.

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[2] The great mathematician Karl Friedrich Gauss discovered this when he was just 10 years old! Check out *Stewart pA35* for the discussion.

Arithmetic Progressions (APs)

An *arithmetic progression* is a sum of the form

$$\boxed{\sum_{k=1}^{n} T_k = a + (a+d) + \cdots + (a + (n-1)d).}$$

where $T_k := a + (k-1)d$, a is the first term and d is the *common difference*. d is found by subtracting *any* two successive terms, i.e. $\boxed{d = T_k - T_{k-1}}$.

So an arithmetic progression is a series where each successive term has a particular difference *added* to it.

Example

$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17$ is an arithmetic progression with first term 1 and common difference of 2.

Writing it in sigma notation:

$$\sum_{i=1}^{9} (1 + (i - 1) \times 2).$$

Let's now derive a general formula for a general arithmetic progression:

Derivation

$$\sum_{k=1}^{n}(a+(k-1)d)$$

$$= a\sum_{k=1}^{n} 1 + d\sum_{k=1}^{n} k - d\sum_{k=1}^{n} 1$$

$$= an + d \times \tfrac{1}{2}n(n+1) - dn$$

$$= \boxed{\tfrac{1}{2}n\left(2a+(n-1)d\right)}$$

$$= n \times \tfrac{1}{2}\left(a+(a+(n-1)d)\right)$$

$$= \boxed{\#\text{ of terms} \times \text{average of first \& last terms}}$$

Determining the number of terms

What if you can't count the number of terms? There is a neat formula we can get from the definition of the AP using the last term.

If there are n terms, and we call the last term ℓ, then clearly we have $\ell := T_n$. And so

$$a+(n-1)d = \ell \iff \boxed{n = \frac{\ell - a}{d} + 1}.$$

Example

Calculate $5 + 9 + 13 + \cdots + 85$

There are 21 terms (check!). Using the previous formula we have the sum $= 21 \times \frac{5+85}{2} = 945.$ Verify for yourselves that using the other formula for the APs also works!

Collapsing Sums

The series of the subtraction of two sequential terms can be written in the following form:

$$\boxed{\sum_{i=F}^{L} [f(i+1) - f(i)]}$$

We call these *collapsing sums* because all the middle terms cancel out and it simplifies nicely!

Let's expand the sigma notation and see what we get:

$$
\begin{aligned}
&(f(F+1)-f(F))\\
&\ +(f(F+2)-f(F+1))\\
&\ +\cdots+(f(L)-f(L-1))\\
&\ +(f(L+1)-f(L))\\
&\ =f(L+1)-f(F)\\
&\ =\boxed{\text{last term } - \text{ first term}}
\end{aligned}
$$

It is a remarkable fact that the sum of the first n odd numbers is equal to n^2, i.e. we have the following series that can be solved using the AP formula:

$$1+3+5+\cdots+(2n-1) = \sum_{k=1}^{n}(2k-1) = n^2$$

Now let us use the trick of collapsing sums to get this result. Notice that $2k-1=k^2-(k-1)^2$, thus

$$\begin{aligned}\sum_{k=1}^{n}[2k-1] &= \sum_{k=1}^{n}\left[k^2-(k-1)^2\right] \\ &= (1^2-0^2)+(2^2-1^2)+(3^2-2^3) \\ &\qquad +\cdots+(n^2-(n-1)^2) \\ &= n^2\end{aligned}$$

Can we use this trick to find

$$\sum_{k=1}^{n} k = 1+2+3\cdots+k,$$

in the same way?

Well from before we have $k = \frac{1}{2}(k^2 - (k-1)^2 + 1)$ and so

$$\sum_{k=1}^{n} k = \tfrac{1}{2}\sum_{k=1}^{n}\left(k^2 - (k-1)^2 + 1\right)$$

$$= \tfrac{1}{2}\sum_{k=1}^{n}\left(k^2 - (k-1)^2\right) + \tfrac{1}{2}\sum_{k=1}^{n}(1)$$

$$= \tfrac{1}{2}n + \tfrac{1}{2}\left(1^2 - 0^2\right) + \tfrac{1}{2}\left(2^2 - 1^2\right)$$

$$+ \tfrac{1}{2}\left(3^2 - 2^2\right) + \dots$$

$$+ \tfrac{1}{2}\left(n^2 - (n-1)^2\right)$$

$$= \tfrac{1}{2}(n + n^2 - 0^2)$$

$$= \tfrac{1}{2}n\,(n+1)$$

Thus we have derived a formula for the sum of the first n natural numbers[3]:

$$\boxed{\sum_{k=1}^{n} k = \tfrac{1}{2}n(n+1).}$$

Then as before we can sum any arithmetic progression!

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[3] *Natural numbers* are positive integers $0, 1, 2, 3, \ldots$

Can we derive the sum of the first n perfect squares $1, 4, 9, 16, 25, \ldots, n^2$? First note that

$$k^3 - (k-1)^3 = 3k^2 - 3k + 1.$$

So $k^2 = \frac{1}{3}[k^3 - (k-1)^3 + 3k - 1]$ and

$$
\begin{aligned}
\sum_{k=1}^{n} k^2 &= \tfrac{1}{3}\sum_{k=1}^{n}\left(k^3-(k-1)^3+3k-1\right) \\
&= \tfrac{1}{3}\sum_{k=1}^{n}\left(k^3-(k-1)^3\right)+\sum_{k=1}^{n}k-\tfrac{1}{3}\sum_{k=1}^{n}1 \\
&= \tfrac{1}{2}n(n+1)-\tfrac{1}{3}n+\tfrac{1}{3}\left(1^3-0^3\right) \\
&\quad +\tfrac{1}{3}\left(2^3-1^3\right)+\cdots+\tfrac{1}{3}\left(n^3-(n-1)^3\right) \\
&= \tfrac{1}{2}n(n+1)-\tfrac{1}{3}n+\tfrac{1}{3}n^3-\tfrac{1}{3}0^3 \\
&= \tfrac{1}{6}n\left(3n+3-2+2n^2\right) \\
&= \tfrac{1}{6}n(n+1)(2n+1).
\end{aligned}
$$

It follows that

$$\boxed{\sum_{k=1}^{n} k^2 = \tfrac{1}{6}n(n+1)(2n+1).}$$

Then, as before, we can sum any series of the form

$$\sum_{k=1}^{n}(ak^2 + bk + c).$$

Finally on this line:

$$k^4 - (k-1)^4 = 4k^3 - 6k^2 + 4k - 1.$$

So $k^3 = \frac{1}{4}[k^4 - (k-1)^4 + 6k^2 - 4k + 1]$ and

$$\begin{aligned}\sum_{k=1}^{n} k^3 &= \tfrac{1}{4}\sum_{k=1}^{n}\Big(k^4-(k-1)^4+6k^2-4k+1\Big)\\
&= \tfrac{1}{4}\sum_{k=1}^{n}\Big(k^4-(k-1)^4\Big)\\
&\quad + \tfrac{3}{2}\sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k + \tfrac{1}{4}\sum_{k=1}^{n} 1\\
&= \tfrac{1}{4}n(n+1)(2n+1) - \tfrac{1}{2}n(n+1) + \tfrac{1}{4}n\\
&\quad + \tfrac{1}{4}\left(1^4-0^4\right) + \tfrac{1}{4}\left(2^4-1^4\right)\\
&\quad + \tfrac{1}{4}\left(3^4-2^4\right) + \cdots + \tfrac{1}{4}\Big(n^4-(n-1)^4\Big)\\
&= \tfrac{1}{4}n(n+1)(2n+1)\\
&\quad - \tfrac{1}{2}n(n+1) + \tfrac{1}{4}n + \tfrac{1}{4}n^4\\
&= \tfrac{1}{4}n^2(n+1)^2.\end{aligned}$$

We obtain the remarkable relation

$$\boxed{\begin{aligned}&\sum_{k=1}^{n} k^3\\ &\quad= \tfrac{1}{4}n^2(n+1)^2\\ &\quad= \left[\tfrac{1}{2}n(n+1)\right]^2\\ &\quad= \left[\sum_{k=1}^{n} k\right]^2.\end{aligned}}$$

Then, as before, we can sum any series of the form

$$\sum_{k=1}^{n}(ak^3+bk^2+ck+d).$$

Geometric Progressions (GPs)

A *geometric progression* is a series where each successive term is *multiplied* by a *common ratio* and then added up. They are of the form

$$\boxed{\sum_{k=1}^{n} T_k = 1 + r + r^2 + \ldots\ + r^{n-1}.}$$

where $T_k := r^{k-1}$ and r is called the *common ratio*. r can be found by dividing *any* two sequential terms,

i.e. $\boxed{r = \dfrac{T_k}{T_{k-1}}}$.

So geometric progressions are series where successive terms have a particular common ratio *multiplied* to them.

Let's derive a formula for a GP:

Derivation

$$\sum_{k=1}^{n} r^{k-1} = 1 + r + r^2 + \ldots + r^{n-1}$$

$$r\left(\sum_{k=1}^{n} r^{k-1}\right) = r + r^2 + r^3 + \ldots + r^n$$

Subtracting we get

$$(1-r)\left(\sum_{k=1}^{n} r^{k-1}\right) = 1 - r^n.$$

Thus, we have

$$\boxed{\sum_{k=1}^{n} r^{k-1} = \frac{1-r^n}{1-r} = \frac{r^n-1}{r-1}.}$$

In a similar way to the APs, we can find the number of terms by noting that the last term is

$$\ell = r^{n-1} \iff \boxed{n = \frac{\log \ell}{\log r} + 1}.^{4}$$

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[4] Here I've written "log" because you can use *any* logarithm, due to the change of base law!

Example

$1+2+4+8+\cdots+4096$ is a geometric progression with common ratio of 2 and 13 terms (check!). Hence

$$\text{Sum } = \frac{2^{13}-1}{2-1} = 8191.$$

Applications of GPs to financial problems

Example

$1000 is invested at the beginning of 1994 and $50 is added to the investment at the beginning of each year for the next 19 years. Interest is paid at 8% per annum in December of each year. What is the total investment at the end of 20 years?

Beginning of year 1

$$1000$$

End of year 1

$$1000 \times (1 + 0.08)$$

$$= 1000 \times 1.08$$

Beginning of year 2

$$1000 \times 1.08 + 50$$

End of year 2

$$(1000 \times 1.08 + 50) \times 1.08$$
$$= 1000 \times (1.08)^2 + 50 \times 1.08$$

Beginning of year 3

$$1000 \times (1.08)^2 + 50 \times 1.08 + 50$$

End of year 3

$$
\begin{aligned}
&(1000 \times (1.08)^2 + 50 \times 1.08 + 50) \times 1.08 \\
&\qquad = 1000 \times (1.08)^3 + 50 \times (1.08)^2 + 50 \times (1.08)
\end{aligned}
$$

Beginning of year 4

$$1000 \times (1.08)^3 + 50 \times (1.08)^2 + 50 \times (1.08) + 50$$

End of year 4

$$
\begin{aligned}
&1000 \times (1.08)^4 + 50 \times (1.08)^3 + 50 \times (1.08)^2 \\
&\qquad + 50 \times 1.08
\end{aligned}
$$

. . .

End of year 20

$$
\begin{aligned}
&1000 \times (1.08)^{20} \\
&\qquad + 50 \times (1.08)^{19} + 50 \times (1.08)^{18} + \\
&\qquad \cdots + 50 \times (1.08)^{2} + 50 \times 1.08 \\
&\quad = 1000 \times (1.08)^{20} + 50 \times 1.08 \times \frac{(1.08)^{19} - 1}{1.08 - 1} \\
&\quad = 6899
\end{aligned}
$$

The investment has grown to \$6899.

Example

Now suppose that the investor wishes to have the money paid out in six equal installments: each installment is to be paid at the beginning of the year; payment starts immediately. Assuming that interest continues to be paid at the rate of 8% per annum, find the amount of each installment.

Let $\$A$ be the amount paid out at the beginning of each year.

End of year 20

$$6899$$

Beginning of year 21

$$6899 - A$$

End of year 21

$$(6899 - A) \times 1.08 \\ = 6899 \times 1.08 - A \times 1.08$$

Beginning of year 22

$$6899 \times 1.08 - A \times 1.08 - A$$

End of year 22

$$(6899 \times 1.08 - A \times 1.08 - A) \times 1.08 \\ = 6899 \times (1.08)^2 - A \times (1.08)^2 - A \times 1.08$$

Beginning of year 23

$$6899 \times (1.08)^2 - A \times (1.08)^2 - A \times 1.08 - A \\ = 6899 \times (1.08)^2 - A \times \big((1.08)^2 + 1.08 + 1\big)$$

Beginning of year 24

$$6899 \times (1.08)^3 - A \times \big((1.08)^3 + (1.08)^2 + 1.08 + 1\big)$$

. . .

Beginning of year 26

$$6899 \times (1.08)^5 - A \times \big((1.08)^5 + \cdots + 1.08 + 1\big)$$

Notice the part in the second brackets is a GP.

At the beginning of year 26 investment is down to \$0. So we have[5]

$$6899 \times (1.08)^5 - A \times \big((1.08)^5 + \cdots + 1\big) = 0$$

$$6899 \times (1.08)^5 = A \times \big((1.08)^5 + \cdots + 1\big)$$

$$6899 \times (1.08)^5 = A \times \frac{(1.08)^6 - 1}{1.08 - 1}$$

$$A = 6899 \times (1.08)^5 \times \frac{1.08 - 1}{(1.08)^6 - 1} \approx 1382$$

Hence each installment is about \$1382.

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[5] Interestingly $6 \times 1382 = 8291 > 6899$, showing that the interest continues to grow the investment if it is withdrawn slowly.

Example

Suppose that \$1000 is invested at 10% and that interest is added:

(a) annually,

(b) monthly,

(c) daily

(d) hourly

(e) continuously.

What is the investment at the end of one year?

(a) $1000 \times 1.1 = 1100.$

(b) $1000 \times \left(1 + \frac{10}{12 \times 100}\right)^{12} = 1104.70.$

(c) $1000 \times \left(1 + \frac{10}{365 \times 100}\right)^{365} = 1105.10.$

(d) $1000 \times \left(1 + \frac{10}{365 \times 24 \times 100}\right)^{365 \times 24}$
$= 1105.17.$

(e) $\lim_{n\to\infty} 1000 \times \left(1 + \frac{10}{n \times 100}\right)^{n} =$
$1105.17.$

So it doesn't matter how often you choose to compound your interest, there is a *limit* to how much an investment can grow!

Discovery!

Suppose that \$1 is invested at 100% interest that is compounded continuously. Then the amount of the investment after one year is actually the constant e.

$$\lim_{n\to\infty} 1\times\left(1+\frac{100}{n\times 100}\right)^n = e \approx 2.71828.$$

Infinite Sums

The final part of this subsection is about *infinite* sums of geometric progressions. We want to calculate

$$\lim_{n\to\infty}\sum_{k=1}^{n} r^{k-1} = \lim_{n\to\infty}\frac{1-r^n}{1-r}.$$

The only time this will yield a finite answer is when $|r| < 1$, because then each successive term becomes increasingly smaller as k gets large.

Indeed it can be shown that in this case $\lim_{n\to\infty} r^n = 0$, and hence we obtain a beautifully simple formula for an infinite GP:

$$\boxed{\sum_{k=1}^{\infty} r^{k-1} = \frac{1}{1-r}.}$$

Example

Sum up all of the inverse binary numbers.

i.e. we want to find $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$

In sigma notation, this is written as

$$\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{k-1}.$$

Hence $r=\frac{1}{2}$ (and indeed $|r|<1$ so the sum will work) and we have

$$\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{k-1}=\frac{1}{1-\frac{1}{2}}=2.$$

Example

Write the recurring decimal

$0.123123123\ldots$ as a fraction.

$$\begin{aligned}
&0.123123123\ldots\\
&=\frac{123}{10^3}+\frac{123}{10^6}+\frac{123}{10^9}+\ldots\\
&=\frac{123}{10^3}\left[1+\frac{1}{10^3}+\frac{1}{10^6}+\ldots\right]\\
&=\frac{123}{10^3}\left[\frac{1}{1-\frac{1}{10^3}}\right]\\
&=\frac{123}{1000}\times\frac{1000}{999}=\frac{123}{999}.
\end{aligned}$$

Here are some interesting formulas using infinite series (neither AP nor GP) for π and e:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}.$$

$$\sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots = e.$$

Indeed infinite series are very common in maths and calculus, and we will see some applications in the next topic: Integration!