Single Variable Integral Calculus

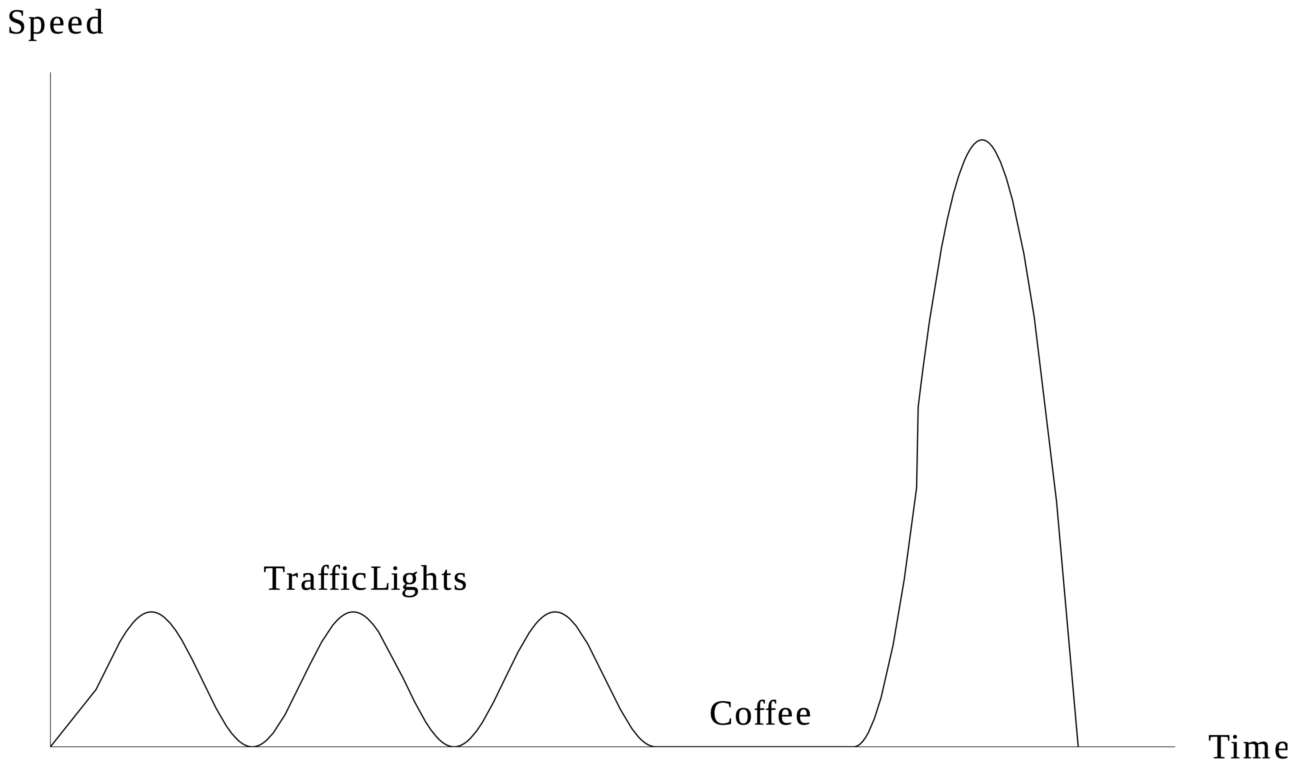
[§§*3.2-3.5 & Appendix 6 of Notes; Chapters 5, 6 & 8 of Stewart*]

In this section, we shall look at problems which lead to definite integrals. We shall work out approximations to definite integrals. We shall visualise the definite integral as "area under the curve." We shall use the fundamental theorem of the calculus to compute definite integrals: the fundamental theorem says, in essence, that **integration is the inverse operation of differentiation**.

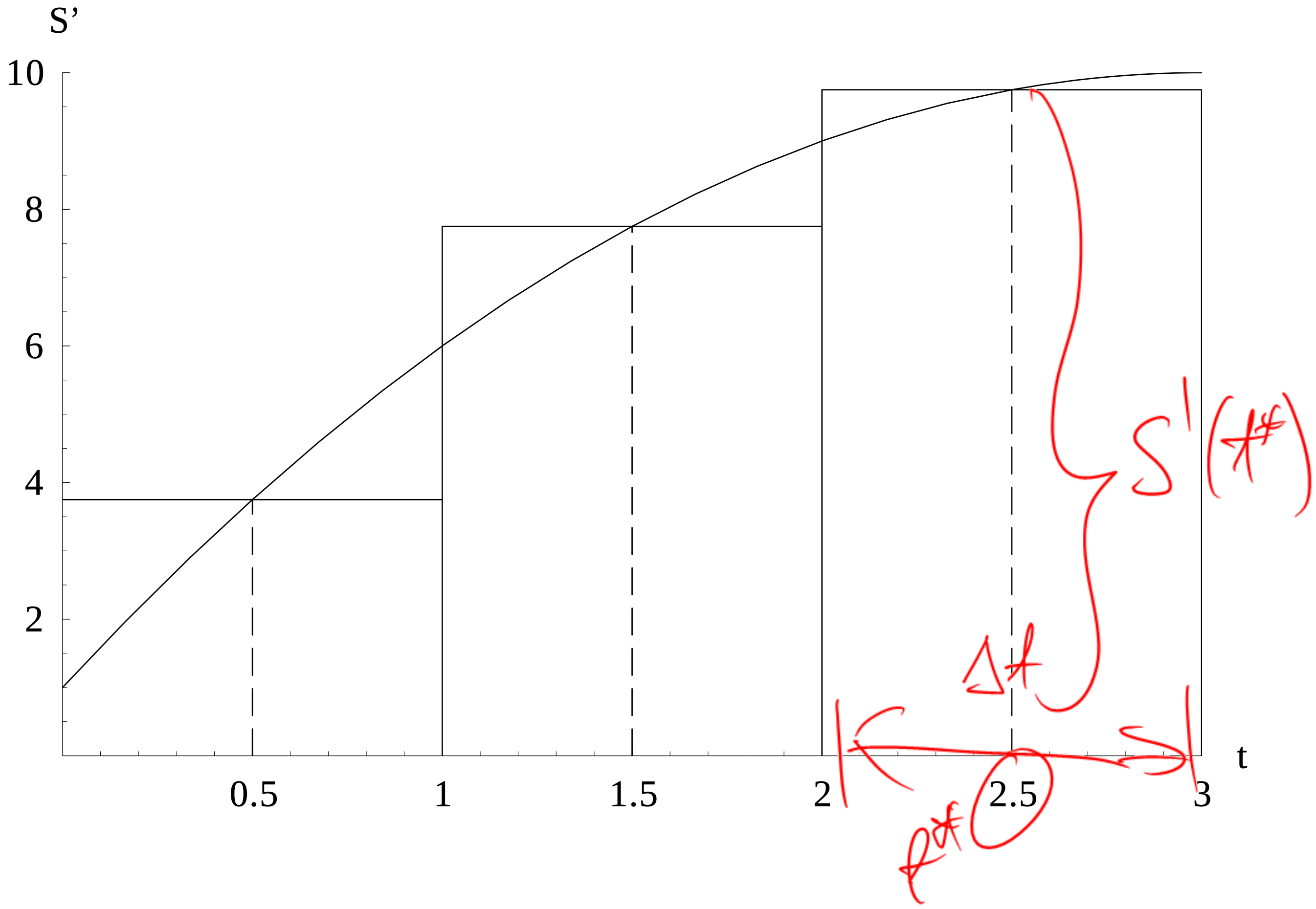
Consider the following problem. You have taken a car trip into country New South Wales: you have an accurate record of your speed at every instant time; but you don't know how far you traveled. How do you find the distance traveled[1]?

If you had traveled at a constant $60\,\mathrm{km}\cdot\mathrm{h}^{-1}$ for three hours, there is no problem: you will have traveled $60 \times 3 = 180$ kilometres. But if the graph of your instantaneous speed looks like the following then have a more difficult problem.

[1] This is called *"The Distance Problem"* in [*Stewart pp296-297*]



Now let us make life a bit easier by taking a a three hour trip with the speed following a simple curve.



Let $S(t)$ be the distance, in kilometres traveled after t hours. Then from Part 3 the speed is $S'(t)$.

Estimate the distance traveled as follows. Record the speed at regular[2] time intervals $\Delta t_i = 1$ hour some time in each hour t_1^* in the first hour, t_2^* in the second hour and t_3^* in the third hour, then an estimate of the distance traveled would be

$$S'(t_1^*) \times 1 + S'(t_2^*) \times 1 + S'(t_3^*) \times 1.$$

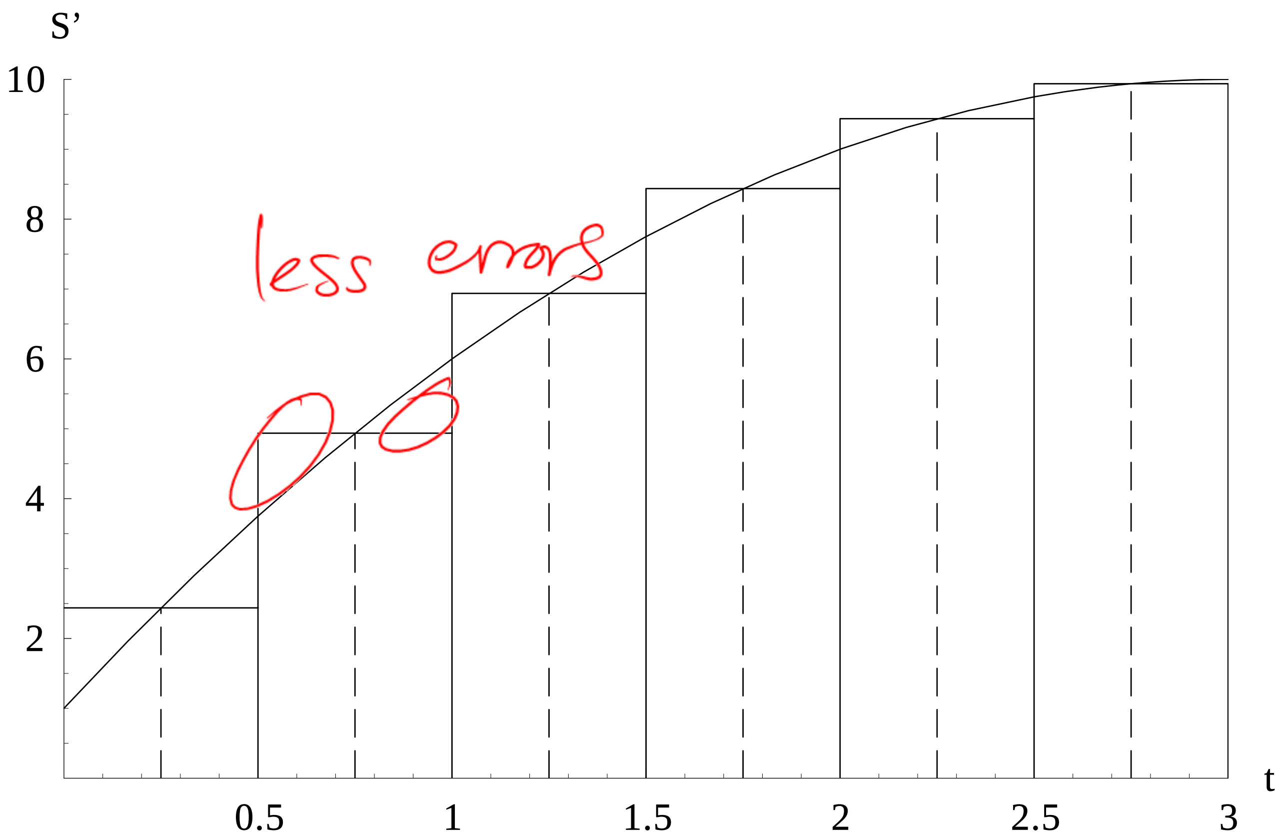
Note this is the sum of the areas of the rectangles, whereby S' are the heights and Δt are the widths.

[2] You may envisage the case whereby you don't take your measurements at regular time intervals. In this course we always assume that our time intervals $\Delta t_i = \Delta t =$ constant. The varying time interval case is studied by advanced students in MATH1903.

Importantly, we can write this in sigma notation as

$$\sum_{i=1}^{3} S'(t_i^*)\Delta t_i.$$

Obviously sampling the speed more frequently (making Δt_i smaller) will get more accurate results.



If we take the "limit" as the thickness of the rectangles get smaller, i.e.

$$\lim_{\Delta t_i \to 0} \Delta t_i = dt,$$

infinitely small but $\neq 0$

we should end up with a completely accurate result.

Clearly the limit (the distance traveled) will turn out to be the "area under the curve." This is because the pointy bits that stick out above and below the curve (so the error) will get smaller and smaller, and the rectangles will fit better and better until a perfect fit is obtained.

Notation

Consider all possible subdivisions of the interval $0 \le t \le 3$:

$$0 = t_0 \le t_1 \le t_2 \le \cdots \le t_n = 3.$$

Take a representative point (denoted by a *) to sit somewhere within each interval, i.e. $t_{i-1} \le t_i^* \le t_i$ and define $\Delta t_i := t_i - t_{i-1}$ to be the particular time interval. Then we take the limit as the subintervals shrink to 0 (or equivalently we take an infinite number of subdivisions) to obtain the *exact* distance traveled:

$$\lim_{n\to\infty} \sum_{i=1}^{n} S'(t_i^*)\Delta t_i.$$

The Definite Integral

This limit is written as:

$$\int_0^3 S'(t)\,dt.$$

It is the *definite integral* of $S'(t)$ over the interval $0 \leq t \leq 3$. We clearly have

Total Change Theorem

$$\int_0^3 S'(t)\,dt = S(3) - S(0)$$

"*The total distance traveled is the difference between the final and initial distances*", and this coincides with our everyday experience.

So we can find the definite integral by finding a function which, when differentiated, gives the speed function S'. → "anti-differentiation"

Now we give the formal definition[3] of the definite integral:

$$\boxed{\int_a^b f(x)\,dx := \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i}$$

where our interval $a \leq x \leq b$ is partitioned as

$$\begin{aligned} a = x_0 &\leq x_1^* \leq x_1 \\ &\leq x_2^* \leq x_2 \leq \dots \\ &\leq x_{n-1} \leq x_n^* \leq x_n = b. \end{aligned}$$

[3] This is the "Reimann Sum" definition.

Later we discover that $\int_a^b f(x)\,dx$ is calculated (as in the speed example) by finding a function whose derivative is f. For the moment we shall content ourselves by estimating values of $\int_a^b f(x)\,dx$.

Since we will only be using subdivisions yielding equal subintervals when making estimates, then given a domain $a \leq x \leq b$ that is divided into n equal subintervals we have

$$\boxed{\Delta x_i = \Delta x_j = \Delta x = \frac{b-a}{n} = \text{constant}}$$

for all choices of i and j.

Example

A bird's speed at a time t hours after midday is $(5-\frac{2}{3}t)\ \mathrm{km}\cdot\mathrm{h}^{-1}$. Estimate the distance traveled from 2-4pm.

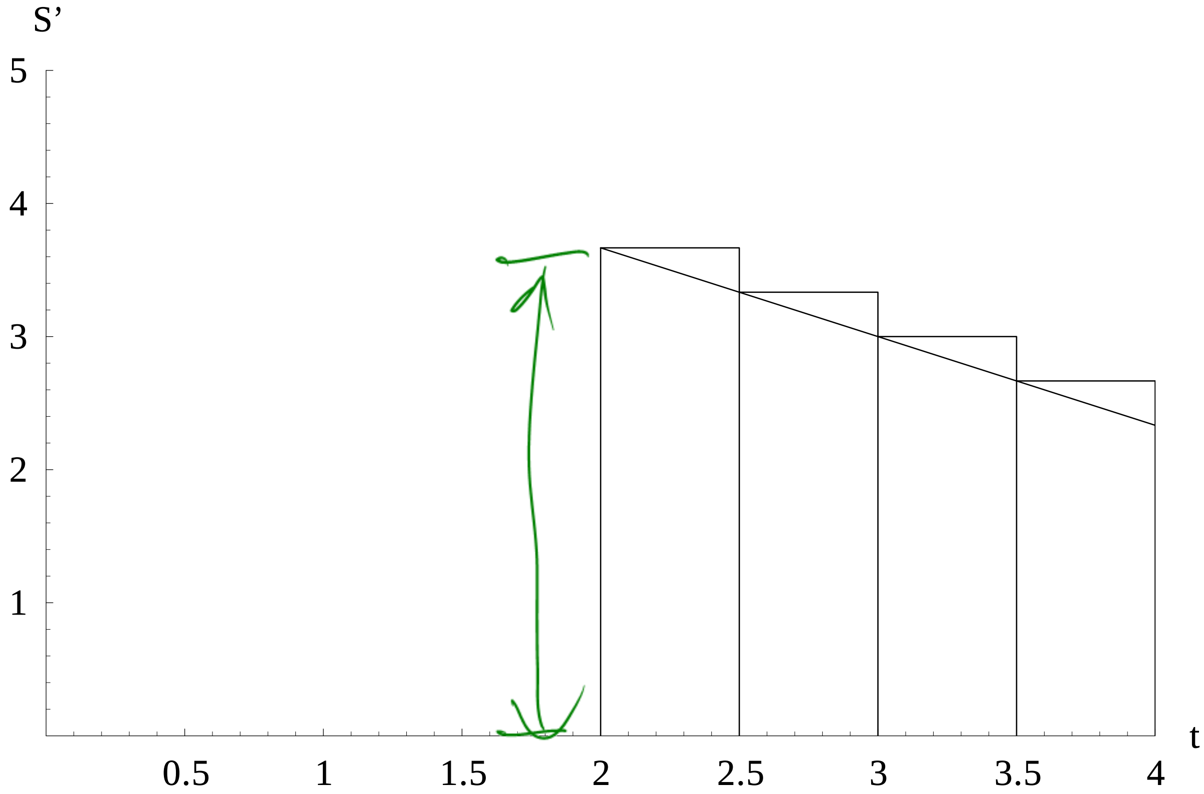
Let the bird travel from $t = 2$ to $t = 4$, and define $S'(t) := 5 - \frac{2}{3}t$. First take 4 subdivisions of the interval:

$$2 \leq 2.5 \leq 3 \leq 3.5 \leq 4$$

where then $\Delta t_i = \frac{4-2}{4} = 0.5$. Let's take our t_i^*s to be the *left hand values*[4]:

$$t_1^* = 2 \quad t_2^* = 2.5 \quad t_3^* = 3 \quad t_4^* = 3.5.$$

[4]MATH1903 students study the more complex cases where the t_i^* is chosen at arbitrary positions within the middles of the intervals.



Heights of rectangles take the left-hand t-values

Estimate is

$$\sum_{i=1}^{4} S'(t_i^*)\Delta t_i$$

$$= \left(5 - \tfrac{2}{3} \times 2\right) \times 0.5$$

$$+ \left(5 - \tfrac{2}{3} \times 2.5\right) \times 0.5$$

$$+ \left(5 - \tfrac{2}{3} \times 3\right) \times 0.5$$

$$+ \left(5 - \tfrac{2}{3} \times 3.5\right) \times 0.5$$

$$= 0.5[5 \times 4 - \tfrac{2}{3}(2 + 2.5 + 3 + 3.5)]$$

$$= 6.33$$

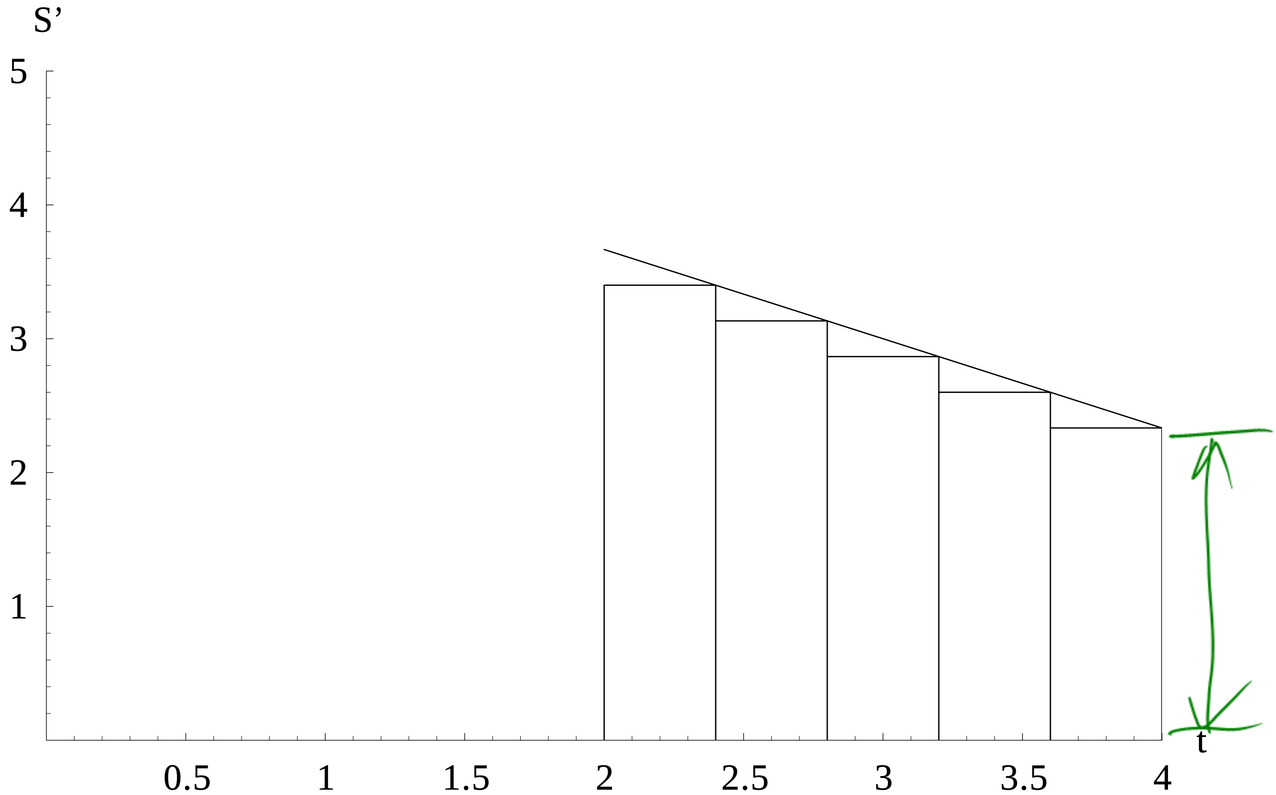
We can make our estimate more accurate by taking 5 subdivisions:

$$2 \leq 2.4 \leq 2.8 \leq 3.2 \leq 3.6 \leq 4$$

whereby now $\Delta t_i = \frac{4-2}{5} = 0.4$. Let's this time[5] take our t_i^*s to be the *right hand values*:

$$t_1^* = 2.4 \quad t_2^* = 2.8 \quad t_3^* = 3.2 \quad t_4^* = 3.6 \quad t_5^* = 4.$$

[5]For the sole purpose of illustration!



Now heights take right-hand t-values

Estimate is

$$\sum_{i=1}^{5} S'(t_i^*)\Delta t_i$$

$$= \left(5 - \tfrac{2}{3} \times 2.4\right) \times 0.4$$

$$+ \left(5 - \tfrac{2}{3} \times 2.8\right) \times 0.4$$

$$+ \left(5 - \tfrac{2}{3} \times 3.2\right) \times 0.4$$

$$+ \left(5 - \tfrac{2}{3} \times 3.6\right) \times 0.4$$

$$+ \left(5 - \tfrac{2}{3} \times 4\right) \times 0.4$$

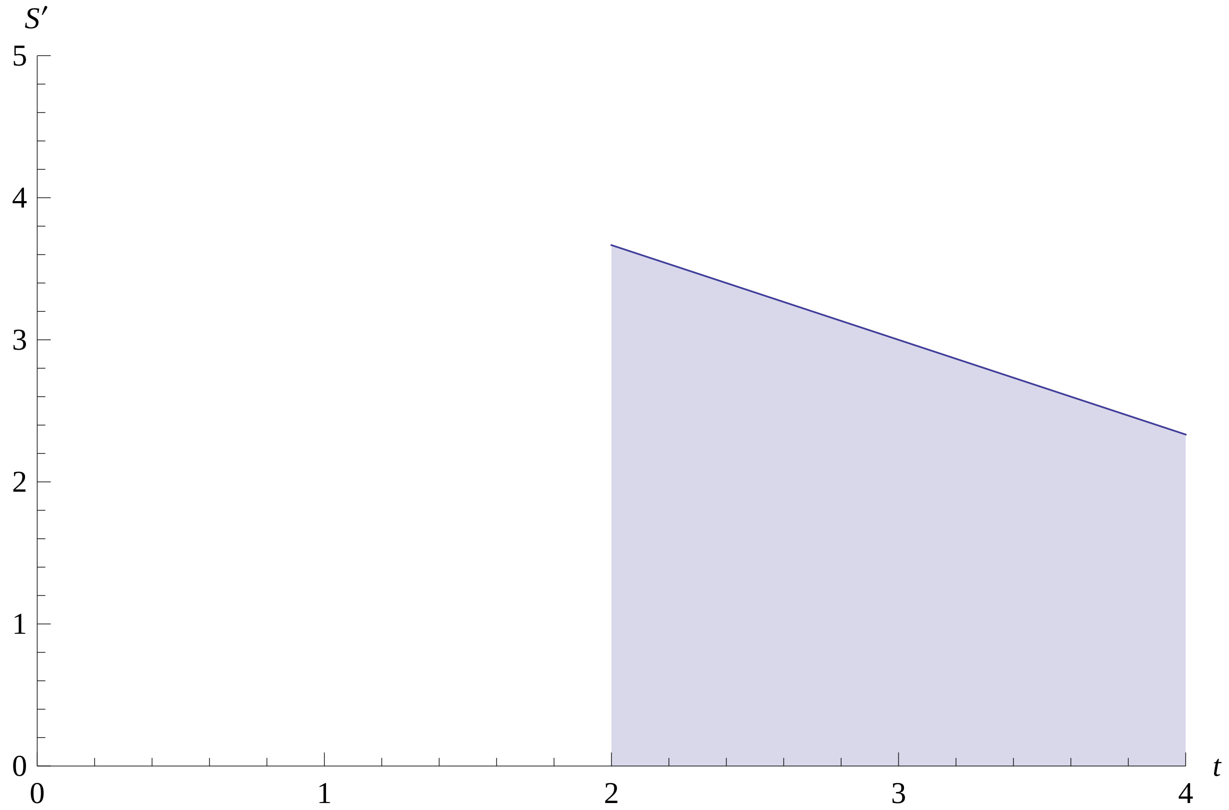
$$= 0.4[5 \times 5$$

$$- \tfrac{2}{3}\left(2.4 + 2.8 + 3.2 + 3.6 + 4\right)]$$

$$= 5.73$$

Let us check our estimates by calculating the exact answer! We know that the exact answer is the integral, namely, the area under the curve (or line in this case). Because our function is a straight line, the area to be found is simply a trapezium. The exact distance traveled then is

$$\begin{aligned} A &= \tfrac{h}{2}(a+b) \\ &= \tfrac{4-2}{2}(S'(4)+S'(2)) \\ &= \left(5-\tfrac{2}{3}\cdot 4\right)+\left(5-\tfrac{2}{3}\cdot 2\right) \\ &= 6. \end{aligned}$$



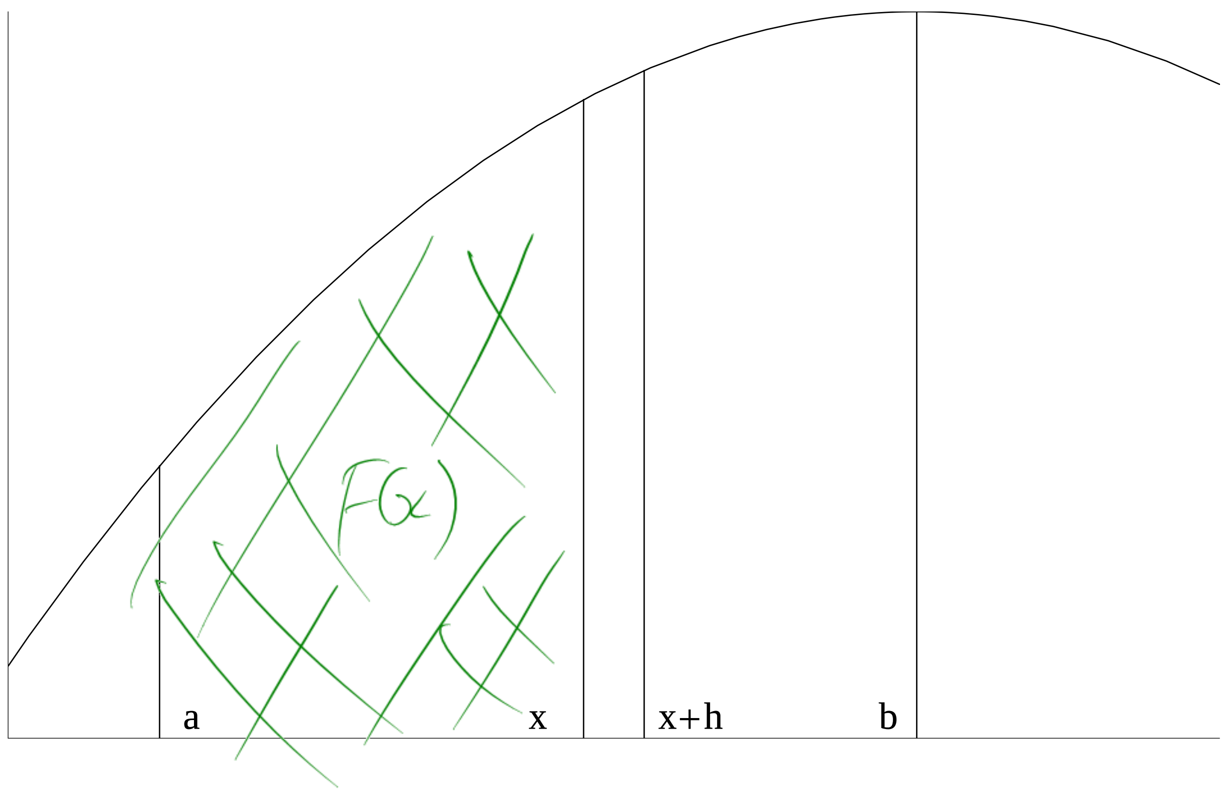
Thus we conclude that the bird has traveled 6 kilometres, and that our second answer performed with more subintervals gave us a more accurate estimate[6].

[6] An error of -0.27 versus $+0.33$.

We have seen that in the case where we are given the speed of a car at every instant we can find the distance traveled (or the area under the curve) by finding a function whose derivative is the speed function. We now see that the integral and the area under the curve can always be found in this way.

Derivation

We will now derive a very important result in calculus which will allow us to calculate integrals.



Consider the curve $y = f(x)$ with interval of integration $a \leq x \leq b$, and let the area under the curve from a to x be $F(x)$. We take a small step h ahead of x. Then the area of the small strip between x and $x + h$ is $F(x + h) - F(x)$.

When h is small we can approximate the area of this strip with a rectangle:

$$\begin{aligned} A &\approx l \times b \\ F(x+h) - F(x) &\approx h \times f(x) \\ f(x) &\approx \frac{F(x+h) - F(x)}{h}. \end{aligned}$$

Taking the limit[7] as $h \to 0$ we get

$$f(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = F'(x).$$

[7] This limit is the *differential calculus from first principles* definition of the derivative that you may have learned or derived in high school.

So to find the area under the curve we need a function F whose derivative is f. The area under the curve from a to b is then

$$\boxed{\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a).}$$

A function F such that $F' = f$ is called the *anti-derivative* or *primitive* of f. The above boxed formula is a version of one of the most important theorems in mathematics: it's called

The Fundamental Theorem of Calculus[8]

[8] See [*p318 in Stewart, p57 of the Notes*].

Here is why the theorem works:

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_a^b F'(x)\,dx \\ &= \int_a^b \frac{dF}{dx}\,dx \\ &= \int_a^b dF \\ &= [F(x)]_a^b \\ &= F(b) - F(a).\end{aligned}$$

"*The integral f' is f*", i.e. we have shown that "*integration is the inverse of differentiation*".

The Indefinite Integral

So far we have only spoken about definite integrals which contain *limits of integration*. They can be used to solve the "Distance Problem", or to find the area under a curve over a given domain.

However sometimes we refer to integrating without specifying the endpoints. We write this as $\int f(x)\,dx$. When we are simply trying to "find a function F such that $F' = f$", i.e. when we just want to find the *anti-derivative* or *primitive* of f, then we are finding the *indefinite integral*.

Note that the result of a definite integral is a *number*, whereas result of an indefinite integral is a *function*.

This motivates us to write out **The Fundamental Theorem of Calculus**[9] in a way that does not involve limits of integration, namely, using an indefinite integral. From our previous work, we have

$$\boxed{\int f'(x)\,dx = f(x) + C}$$

where C is called the "constant of integration".

[9] A similar, but more advanced, representation of this is given in [*Stewart p315*].

Why do we need this "$+C$"?! Well, suppose F is an anti-derivative of f, namely $F' = f$. If we add a constant to F we get the same answer! Because, if $G := F + C$, is another anti-derivative, where C is an arbitrary *constant of integration*, then $G' = (F + C)' = F' + C' = F' + 0 = F' = f$. Hence G is also a primitive of f. So writing the $+C$ encapsulates *all* possible primitives.

We immediately know lots of indefinite integrals because of our knowledge of differentiation; we simply apply these rules in reverse:

Powers

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n \qquad n \neq -1$$

$$\boxed{\Rightarrow \int x^n \, dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1}$$

$$\frac{d}{dx}[\ln x] = \frac{1}{x} \qquad x > 0$$

$$\frac{d}{dx}[\ln(-x)] = \frac{1}{-x} \times (-1) = \frac{1}{x} \qquad x < 0$$

$$\boxed{\Rightarrow \int x^{-1} \, dx = \ln|x| + C \qquad x \neq 0}$$

This integral can also be written as

$$\int \frac{1}{x} \, dx = \int \frac{dx}{x}.$$

Exponentials

$$\frac{d}{dx}\left[e^x\right] = e^x$$

$$\boxed{\Rightarrow \int e^x\,dx = e^x + C}$$

Trigonometric functions

$$\frac{d}{dx}\left[\sin x\right] = \cos x$$

$$\frac{d}{dx}\left[\cos x\right] = -\sin x$$

$$\boxed{\Rightarrow \begin{array}{l} \displaystyle\int \sin x\,dx = -\cos x + C \\ \displaystyle\int \cos x\,dx = \sin x + C \end{array}}$$

Linearity results

$$\frac{d}{dx}\left[kf(x)\right] = k\frac{d}{dx}f(x)$$

$$\frac{d}{dx}\left[f(x)+g(x)\right] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

$$\int kf(x)\,dx = k\int f(x)\,dx$$

$$\Rightarrow \int \left[f(x)+g(x)\right]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

Also look at [*Appendix 6 of the Notes*] for a brief table of integrals.

Example

$$\int \left(3x^5 - 7x^2 + 2\right)\, dx$$

$$= \tfrac{1}{2}x^6 - \tfrac{7}{3}x^3 + 2x + C$$

Example

$$\int \frac{t^2 - 5}{t^3}\, dt \qquad (t \neq 0)$$

$$= \int \left(t^{-1} - 5t^{-3}\right)\, dt$$

$$= \ln|t| - 5 \times \frac{t^{-2}}{(-2)} + C$$

$$= \ln|t| + \frac{5}{2t^2} + C$$

Example

$$\int \sqrt{x}(2-x)\,dx$$

$$= \int \left(2x^{\frac{1}{2}} - x^{\frac{3}{2}}\right)\,dx$$

$$= \frac{2x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + \frac{x^{\frac{5}{2}}}{\left(\frac{5}{2}\right)} + C$$

$$= \tfrac{4}{3}x\sqrt{x} - \tfrac{2}{5}x^2\sqrt{x} + C$$

$$= 2x\sqrt{x}\left(\frac{2}{3} - \frac{1}{5}x\right) + C$$

$$= \frac{2}{15}x\sqrt{x}\,(10-3x) + C$$

Example

$$\int 5e^x\,dx = 5\int e^x\,dx = 5e^x + C$$

Example

The slope of a curve at x is $2x + 1$ and the curve passes through $(1, 5)$. What is the equation of the curve?

Use FTC! We let

$$f'(x) = 2x + 1.$$

$$\implies f(x) = \int (2x+1)\,dx = x^2+x+C.$$

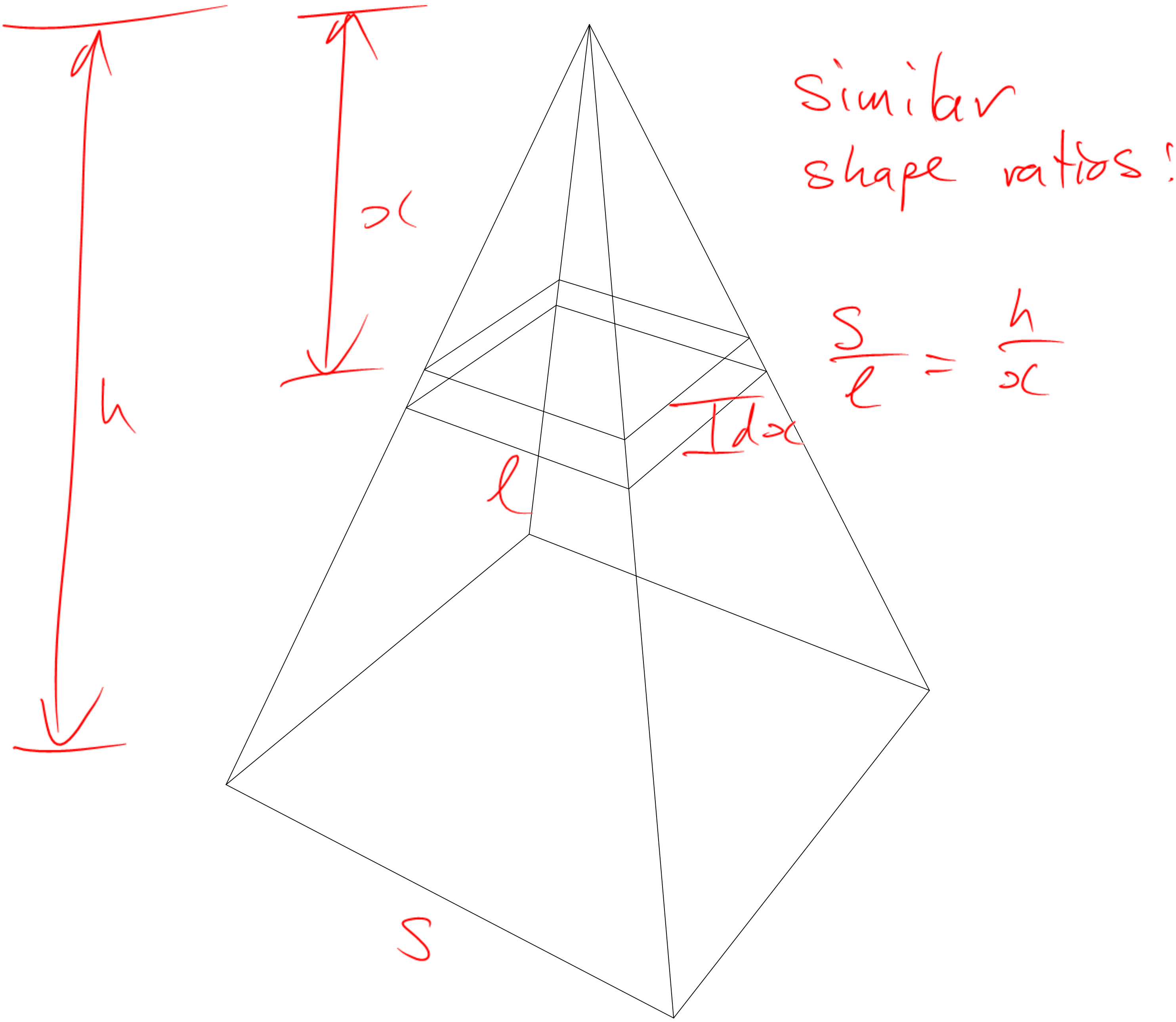
Since $f(1) = 5$, we have $1+1+C = 5 \implies C = 3$. So the curve is

$$y = x^2 + x + 3.$$

Application of definite integrals

Example

Find the volume a pyramid with a square base of side s and height h.



To find the volume we cut into thin horizontal slices and add up the volumes of the slices. This turns the problem into an integral evaluation.

Take a slice a distance x down from the apex: using properties of similar shapes we can show that we get a slice which is a square of side

$$s \times \frac{x}{h}.$$

The volume of this slice is

$$\Delta V = \left(\frac{s}{h}\right)^2 x^2 \, \Delta x.$$

Now letting the thickness of the slices get infinitesimally small ($\Delta x \to dx$) we get

$$dV = \left(\frac{s}{h}\right)^2 x^2 \, dx.$$

Adding up all the thin slices (i.e. integrating from $x = 0$ to h) gives us

$$
\begin{aligned}
V &= \frac{s^2}{h^2}\int_0^h x^2\,dx \\
&= \frac{s^2}{h^2}\left[\frac{x^3}{3}\right]_0^h \\
&= \tfrac{1}{3}s^2 h \\
&= \tfrac{1}{3}\times \text{Area of base} \times \text{height.}
\end{aligned}
$$

I think it's pretty neat that you can use calculus to derive those weird and wonderful geometry formulas that you took for granted way back in junior high school!

Example

Suppose that the rate of growth of a population is given by the differential equation

$$\frac{dP}{dt} = 100e^t,$$

where P is the population at time t years after some starting date. How much will the population increase in the first 10 years?

$$
\begin{aligned}
\text{Increase} &= P(10) - P(0) \\
&= [P(t)]_0^{10} \\
&= \int_0^{10} \frac{dP}{dt}\, dt \\
&= 100 \int_0^{10} e^t \, dt \\
&= 100 \left[e^t \right]_0^{10} \\
&= 100 \left(e^{10} - 1 \right).
\end{aligned}
$$

Example

Suppose that the initial population in the previous example had been 250. Find a formula for $P(t)$.

$$\frac{dP}{dt} = 100e^t \implies P(t) = 100e^t + C.$$

Use the *initial condition* to calculate the *constant of integration* C:

$$\begin{aligned} P(0) &= 250 \\ \implies 250 &= 100 \times e^0 + C \\ \iff C &= 150 \\ \implies P(t) &= 200e^t + 150. \end{aligned}$$

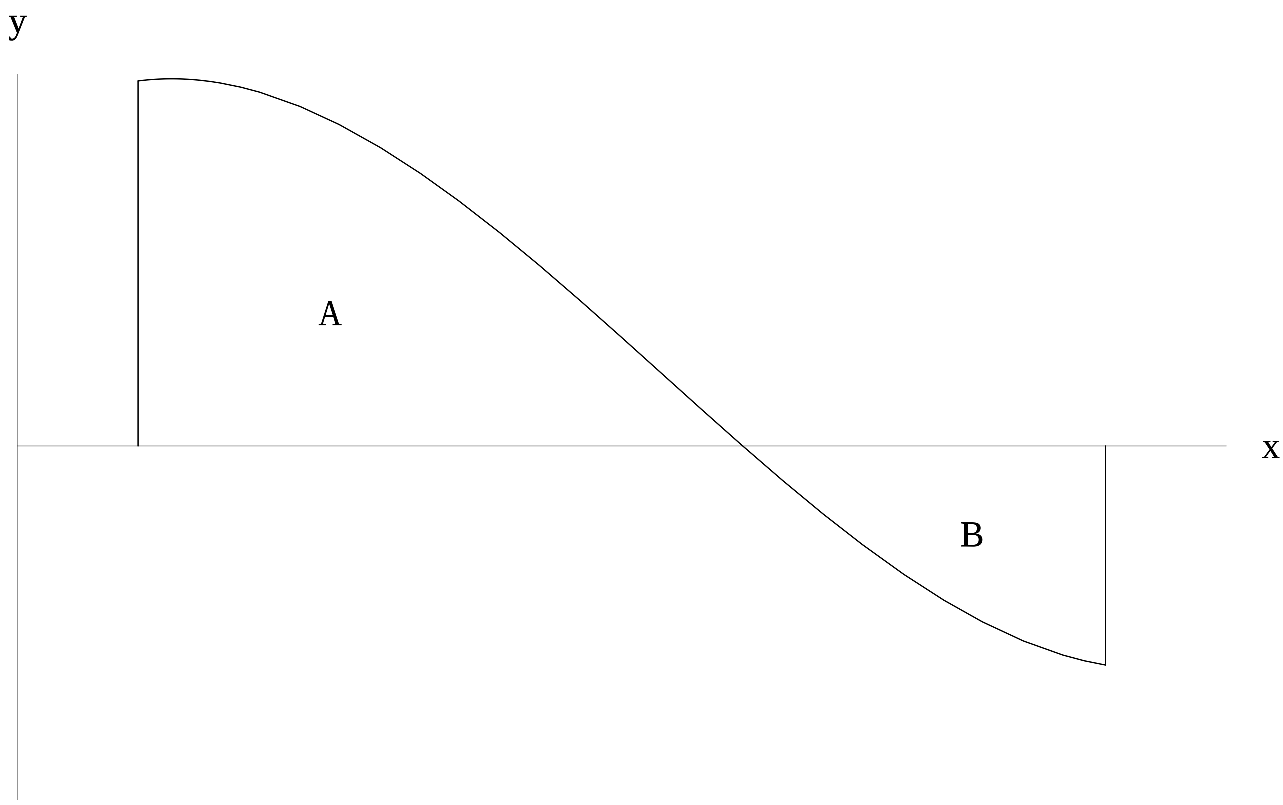
There is a fairly "unorthodox" method of integrating, but if you like it then you can use it. You can "separate" the differential variables[10] and integrate both sides to give

$$\begin{aligned} \frac{dP}{dt} &= 100e^t \\ \iff dP &= 100e^t \, dt \\ \iff \int_{250}^{P} d\hat{P} &= 100 \int_0^t e^{\hat{t}} \, d\hat{t} \\ \iff \left[\hat{P}\right]_{250}^{P} &= 100 \left[e^{\hat{t}}\right]_0^t \\ \iff P - 250 &= 100\left(e^t - e^0\right) \\ \iff P(t) &= 100e^t + 150. \end{aligned}$$

[10] This method is called *Separation of Variables* and is taught in MATH1013.

Algebraic and geometric area

Note that sometimes $\int_a^b$ gives an *algebraic* version of the area under the curve. In the following situation:



The area A is positive (above x-axis) and vice-versa for B.

Therefore the “algebraic” area (or just definite integral) is

$$\int_a^b f(x)\,dx = A - B$$

whilst the “geometric” or “absolute” area is the sum of all the areas taken positively, namely

$$A + B = \int_a^b |f(x)|\,dx.$$

More integration techniques

Integration by parts

[§*8.1 in Stewart*]

This technique is required to solve a question in the assignment. Although this will not be *explicitly* tested in the upcoming quiz or exam, you are free to use it to evaluate any integrals if you so wish. It is studied by Ext. II Math students in NSW and in MATH1003.

We can use the product rule from differentiation to derive the *integration by parts* technique.

Recall the product rule:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

or equivalently (rearranging)

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

and integrating both sides yields:

$$\boxed{\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx}$$

or more "colloquially" using Newton's notation:

$$\boxed{\int uv' = uv - \int u'v.}$$

Look at the formula above. In effect, integration by parts moves a derivative from one function to another. We often use integration by parts to integrate the product of two functions.

Choose a function to differentiate (the "u"), and the other you integrate (the "v'"). Since you can easily differentiate any function, the only restriction is that you cannot integrate all functions. In the event that you can integrate both functions, then you choose the function to differentiate (u) according to the mnemonic LATE:

1	L	**L**ogarithmic	eg. $\ln x$
2	A	**A**lgebraic	eg. $x^2, \sqrt{x}$ etc
3	T	**T**rigonometric	eg. $\sin x$
4	E	**E**xponential	eg. e^x

This is best demonstrated with a couple of examples:

Example

Find $\int xe^x\,dx$.

Using LATE, we choose[11]

$$u = x \implies u' = 1$$

$$v' = e^x \implies v = e^x$$

$$\text{So} \int xe^x\,dx = uv - \int u'v$$

$$= xe^x - \int e^x\,dx$$

$$= xe^x - e^x + C$$

$$= e^x(x-1) + C$$

[11]Note that we can choose *any* anti-derivative for v' so we don't need to include the $+C$.

Example

Evaluate $\int_0^1 xe^x\,dx$.

If we have limits on the integral, we put limits on both the uv part and the $\int u'v$ part:

$$\begin{aligned}\int_0^1 xe^x\,dx &= uv|_0^1 - \int_0^1 u'v \\ &= xe^x|_0^1 - \int_0^1 e^x\,dx \\ &= (1e^1 - 0e^0) - e^x|_0^1 \\ &= e - (e^1 - e^0) \\ &= 1.\end{aligned}$$

Example (tricky!)

Find $\displaystyle\int \ln x \, dx$.

Firstly, write $\ln x = 1 \times \ln x = x^0 \times \ln x$, then using LATE we choose

$$u = \ln x \implies u' = \frac{1}{x}$$

$$v' = x^0 \implies v = x$$

$$\begin{aligned}\int \ln x \, dx &= x \ln x - \int \frac{1}{x} \cdot x \, dx \\ &= x \ln x - \int dx \\ &= x \ln x - x + C \\ &= x(\ln x - 1) + C.\end{aligned}$$

Integration by recognition; Integration by substitution

$[Stewart\ \S 5.5]$

Two useful techniques for integration come out of the chain rule for differentiation: *integration by recognition* and *integration by substitution*.

Let F be a primitive for f: so that $F' = f$.

Let $H(x) = F(g(x))$.

Then

$$H'(x) = F'\left(g(x)\right) g'(x) = f\left(g(x)\right) g'(x).$$

So

$$\begin{aligned}
&\int f\left(g(x)\right) g'(x)\, dx \\
&= \int F'\left(g(x)\right) g'(x)\, dx \\
&= \int H'(x)\, dx \\
&= H(x) + C \\
&= F\left(g(x)\right) + C.
\end{aligned}$$

Hence if F is a primitive (aka anti-derivative) for f then $F(g(x))$ is a primitive for $f(g(x))\,g'(x)$: all you have to do is **recognise** it. So the formula to memorise is:

anti-derivative of f

inside

$$\int f(g(x))\,g'(x)\,dx = F(g(x)) + C.$$

outside

derivative of inside

put inside back in

Here's another way of doing things:
Let $u = g(x)$. Then

$$\begin{aligned}\int f\left(g(x)\right) g'(x)\, dx &= \int f\left(u\right) \frac{du}{\cancel{dx}}\, \cancel{dx} \\ &= \int f\left(u\right)\, du \\ &= F\left(u\right) + C \\ &= F\left(g(x)\right) + C.\end{aligned}$$

So the idea is:

given $\int f\left(g(x)\right) g'(x)\, dx$
substitute $u = g(x)$
and get

$$\int f\left(g(x)\right) g'(x)\, dx = \int f\left(u\right)\, du.$$

In other words: "make a substitution for the inside mess that will get rid of the outside derivative".

So there are two ways of integration using the chain rule:

(a) *integration by recognition* (quicker calculation but harder to understand): "massage" the integrand to make the derivative of the inside function appear multiplied by the outside, and then simply integrate the outside function "as normal", but keep the original inside function inside.

(b) *integration by substitution* (longer calculation but easier to understand): simply make a substitution for the messy inside bit, differentiate, and substitute in the derivative to make it cancel, and then integrate the substituted function.

We can use what we've just seen to prove the formula for integrating functions of linear functions. Namely

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C$$

For example:

$$\int e^{5x-7}\,dx = \frac{1}{5}e^{5x-7} + C$$

Example

Find $\int x(x^2+4)^4\,dx$.

derivative outside inside

First by *recognition*:

$$\int x(x^2+4)^4\,dx$$

$$=\frac{1}{2}\int 2x(x^2+4)^4\,dx$$

$$=\frac{1}{2}\times\frac{(x^2+4)^5}{5}+C$$

$$=\frac{(x^2+4)^5}{10}+C.$$

Now by *subsitution*. We make the substitution

$$\begin{aligned} u &= x^2 + 4 \\ \implies \frac{du}{dx} &= 2x \\ \iff dx &= \frac{du}{2x}. \end{aligned}$$

Then substituting those in we get

$$\int x(x^2+4)^4\,dx$$

$$= \int xu^4\,\frac{du}{2x}$$

$$= \frac{1}{2}\int u^4\,du$$

$$= \tfrac{1}{2} \times \tfrac{1}{5}u^5 + C$$

$$= \tfrac{1}{10}(x^2+4)^5 + C.$$

Example

Find $\displaystyle\int xe^{3x^2+2}\,dx$.

(*Recognition*)

$$\int xe^{3x^2+2}\,dx$$

$$= \frac{1}{6}\int 6xe^{3x^2+2}\,dx$$

$$= \tfrac{1}{6}e^{3x^2+2} + C.$$

(*Substitution*)

$$
\begin{aligned}
u &= 3x^2 + 2 \\
\implies \frac{du}{dx} &= 6x \\
\iff dx &= \frac{du}{6x}.
\end{aligned}
$$

Substituting that in we get

$$\int xe^{3x^2+2}\,dx$$

$$= \int xe^u \frac{du}{6x}$$

$$= \frac{1}{6}\int e^u\,du$$

$$= \tfrac{1}{6}e^u + C$$

$$= \tfrac{1}{6}e^{3x^2+2} + C.$$

Example

Find $\int \sin^3 x \cos x \, dx$

$$\int \sin^3 x \cos x \, dx$$

$$= \int (\sin x)^3 \cos x \, dx$$

$$= \tfrac{1}{4} \sin^4 x + C.$$

or

Substitute $u = \sin x$, so that $dx = \frac{du}{\cos x}$. Then

$$\int \sin^3 x \cos x \, dx$$

$$= \int u^3 \, du$$

$$= \tfrac{1}{4} u^4 + C$$

$$= \tfrac{1}{4} \sin^4 x + C.$$

Example

Sometimes we can just write down the answer in one line!

$$\int (3x^2 + 4x)\sin(x^3 + 2x^2 + 3)\,dx$$

$$= -\cos(x^3 + 2x^2 + 3) + C$$

or

Substitute $u = x^3 + 2x^2 + 3$, so that $dx = \frac{du}{3x^2+4x}$. Then

$$\int (3x^2 + 4x)\sin(x^3 + 2x^2 + 3)\,dx$$

$$= \int \sin u\,du$$

$$= -\cos u + C$$

$$= -\cos(x^3 + 2x^2 + 3) + C.$$

Note: when using the substitution method to integrate an indefinite integral, you must **always** substitute in the original equation for u in terms of x. However this is not necessary when integrating a definite integral, as the next example illustrates.

Example

Find $\int_0^1 x^2 e^{x^3+7}\,dx$.

$$\begin{aligned}
&\int_0^1 x^2 e^{x^3+7}\,dx \\
&= \frac{1}{3}\int_0^1 3x^2 e^{x^3+7}\,dx \\
&= \frac{1}{3}\left[e^{x^3+7}\right]_0^1 \\
&= \frac{1}{3}\left(e^8 - e^7\right) \\
&= \frac{1}{3}e^7\left(e-1\right).
\end{aligned}$$

Now integrating by substitution requires a few more steps. We make the substitution as before, but we must also **change limits**. We demonstrate here:

$$
\begin{aligned}
u &= x^3 + 7 \\
\implies \frac{du}{dx} &= 3x^2 \\
\iff dx &= \frac{du}{3x^2} \\
x = 0 &\implies u = 7 \\
x = 1 &\implies u = 8.
\end{aligned}
$$

Putting all that info together gives

$$\int_0^1 x^2 e^{x^3+7}\,dx$$

$$= \int_7^8 x^2 e^u \,\frac{du}{3x^2}$$

$$= \frac{1}{3}\int_7^8 e^u\,du$$

$$= \frac{1}{3}\, e^u\Big|_7^8$$

$$= \frac{1}{3}\left(e^8 - e^7\right)$$

$$= \frac{1}{3}e^7\,(e-1)$$

and (of course) the same answer!

Example (tricky!)

Find $\int_0^{\frac{\pi}{4}} \tan x \, dx$.

$$\int_0^{\frac{\pi}{4}} \tan x \, dx = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} \, dx$$

$$= -\int_0^{\frac{\pi}{4}} (-\sin x) \frac{1}{\cos x} \, dx$$

$$= -\left[\ln|\cos x|\right]_0^{\frac{\pi}{4}}$$

$$= -\left(\ln\left|\cos\left(\frac{\pi}{4}\right)\right| - \ln|\cos 0|\right)$$

$$= \ln|1| - \ln\left|\frac{1}{\sqrt{2}}\right| = \frac{1}{2}\ln 2.$$

Alternatively by substitution:

$$\begin{aligned} u &= \cos x \\ \implies \frac{du}{dx} &= -\sin x \\ \iff dx &= -\frac{du}{\sin x} \\ x = 0 &\implies u = 1 \\ x = \frac{\pi}{4} &\implies u = \frac{1}{\sqrt{2}}. \end{aligned}$$

Putting all that info together gives

$$\int_0^{\frac{\pi}{4}} \tan x \, dx = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} \, dx$$

$$= -\int_1^{\frac{1}{\sqrt{2}}} \frac{\sin x}{u} \frac{du}{\sin x}$$

$$= -\int_1^{\frac{1}{\sqrt{2}}} \frac{du}{u}$$

$$= -\left[\ln |u|\right]_1^{\frac{1}{\sqrt{2}}}$$

$$= -\left(\ln \left|\frac{1}{\sqrt{2}}\right| - \ln |1|\right) = \frac{1}{2} \ln 2.$$

and (of course) the same answer!

Using integration to average

[*Stewart §6.5, §3.4.3 of Notes*]

At school we learnt how to find the average of a set of *discreet* data points $\{x_1, x_2, \ldots, x_n\}$:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

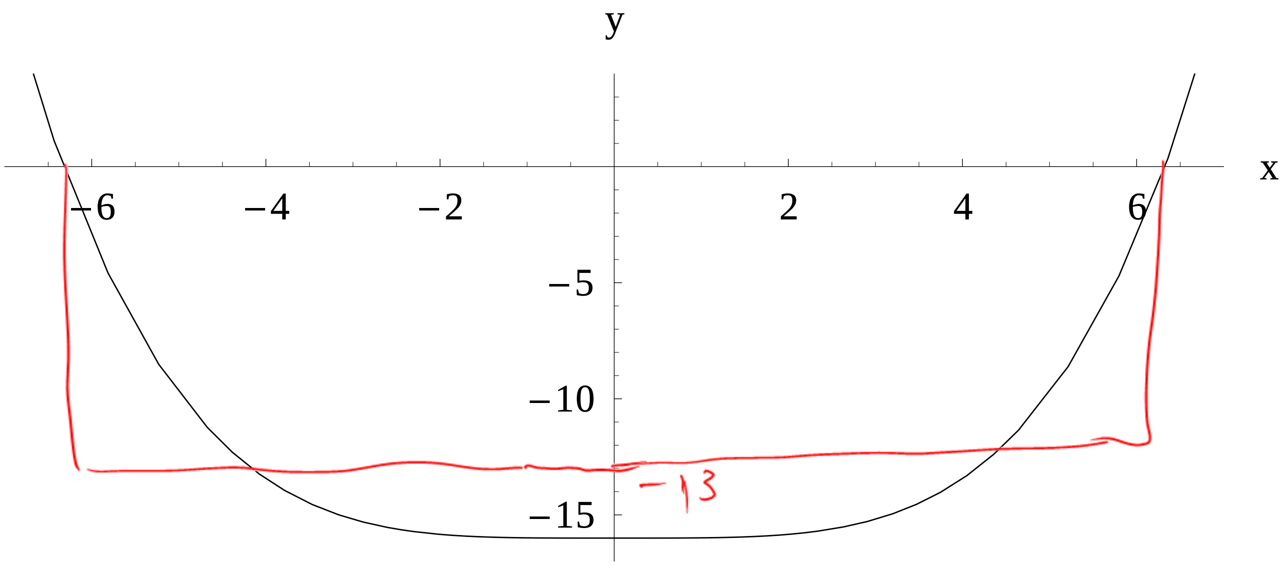
We would now like to find the *continuous* analogy, and for that we use integration. The following example will illustrate.

Example

Find the average depth of a river at a point where the river bed is described by the equation

$$y = 0.01x^4 - 16.$$

The water surface is at $y = 0$. All distances in meters.



Clearly the average depth is the depth of a rectangular channel with the same width as the river and the same cross-sectional area (think about this!). We see that the river stretches from $x = -2\sqrt{10}$ to $x = 2\sqrt{10}$ and thus has width $4\sqrt{10}$. The cross-sectional area is

$$\int_{-2\sqrt{10}}^{2\sqrt{10}} \left(16 - 0.01x^4)\right)\, dx$$

$$= \left[16x - 0.002x^5\right]_{-2\sqrt{10}}^{2\sqrt{10}}$$

$$= 161.909 \approx 162 \text{ m}^2.$$

The average depth of the river (in meters) is approximately

$$\frac{\text{area}}{\text{width}} \approx \frac{162}{4\sqrt{10}} \approx 13 \text{ m.}$$

In general we define the average of the function $f(x)$ over the interval $a \leq x \leq b$ to be

$$\boxed{\frac{1}{b-a}\int_a^b f(x)\,dx.}$$

Example

The number of individuals $P(t)$ in a rabbit population at time t years is given by

$$P(t) = 200e^{0.2t}.$$

Find the average population from $t = 0$ to $t = 10$.

The average is

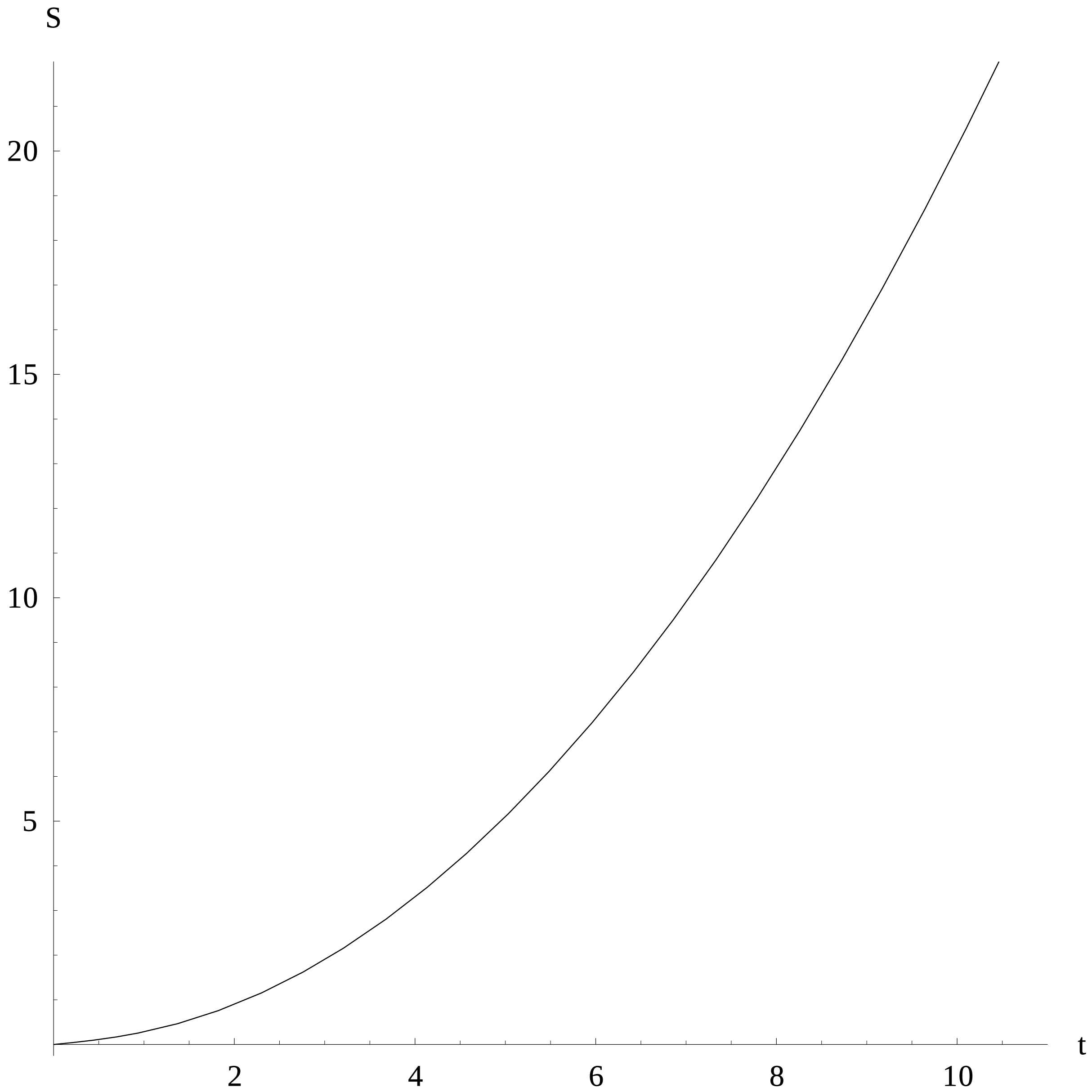
$$\frac{1}{10} \times 200 \int_0^{10} e^{0.2t}\,dt$$

$$= \frac{1}{10} \times \frac{200}{0.2} \left[e^{0.2t}\right]_0^{10}$$

$$= 638.906 \approx 639 \text{ rabbits.}$$

Example

A car travels so that its speed after t seconds is $\frac{1}{5}t\sqrt{1+t^2}$ m/s. Find its average speed over the first 10 seconds.



Average Speed

$$= \frac{1}{10}\int_0^{10} \tfrac{1}{5}t\sqrt{1+t^2}\,dt$$

$$= \frac{1}{50}\int_0^{10} t(1+t^2)^{\frac{1}{2}}\,dt$$

$$= \frac{1}{50}\times\frac{1}{2}\int_0^{10} 2t(1+t^2)^{\frac{1}{2}}\,dt$$

$$= \frac{1}{100}\left[\tfrac{2}{3}(1+t^2)^{\frac{3}{2}}\right]_0^{10}$$

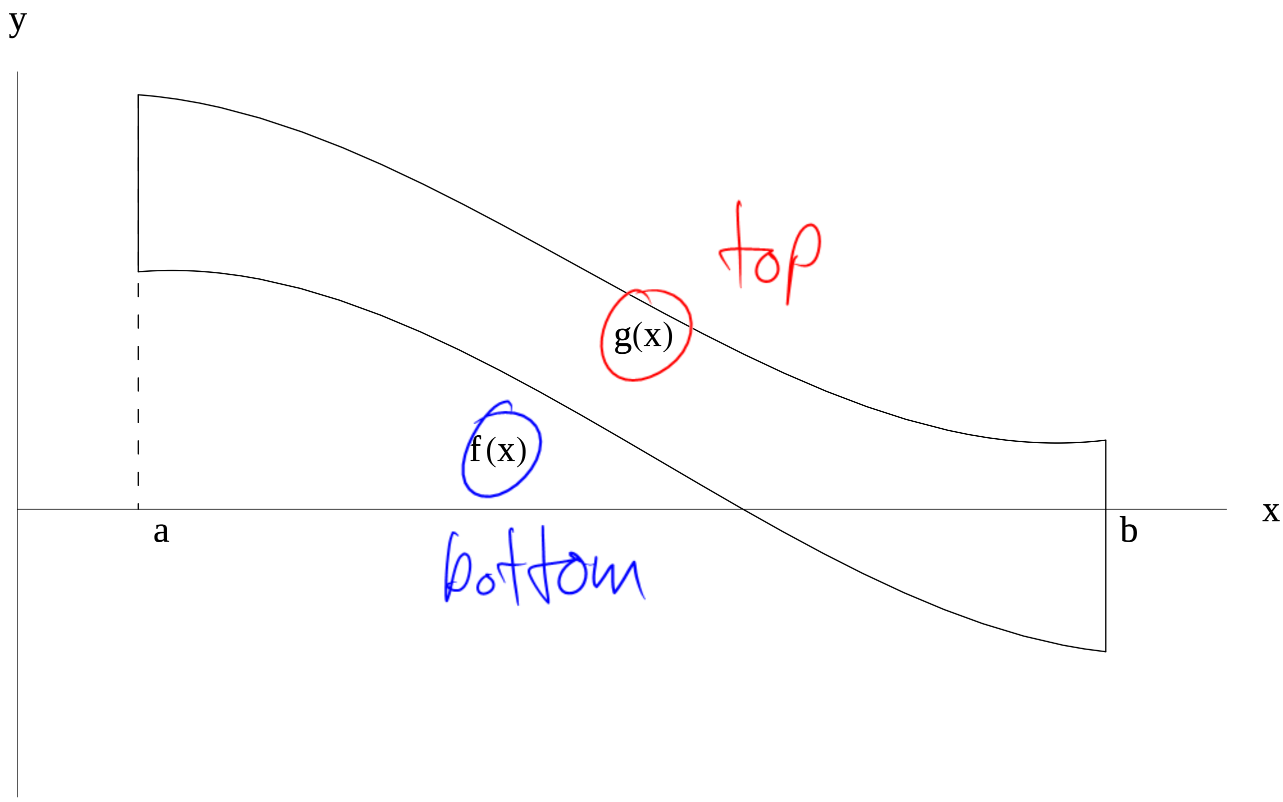
$$= \frac{1}{150}\left((101)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right)$$

$$\approx 6.8 \text{ m}\cdot\text{s}^{-1}.$$

Area between curves

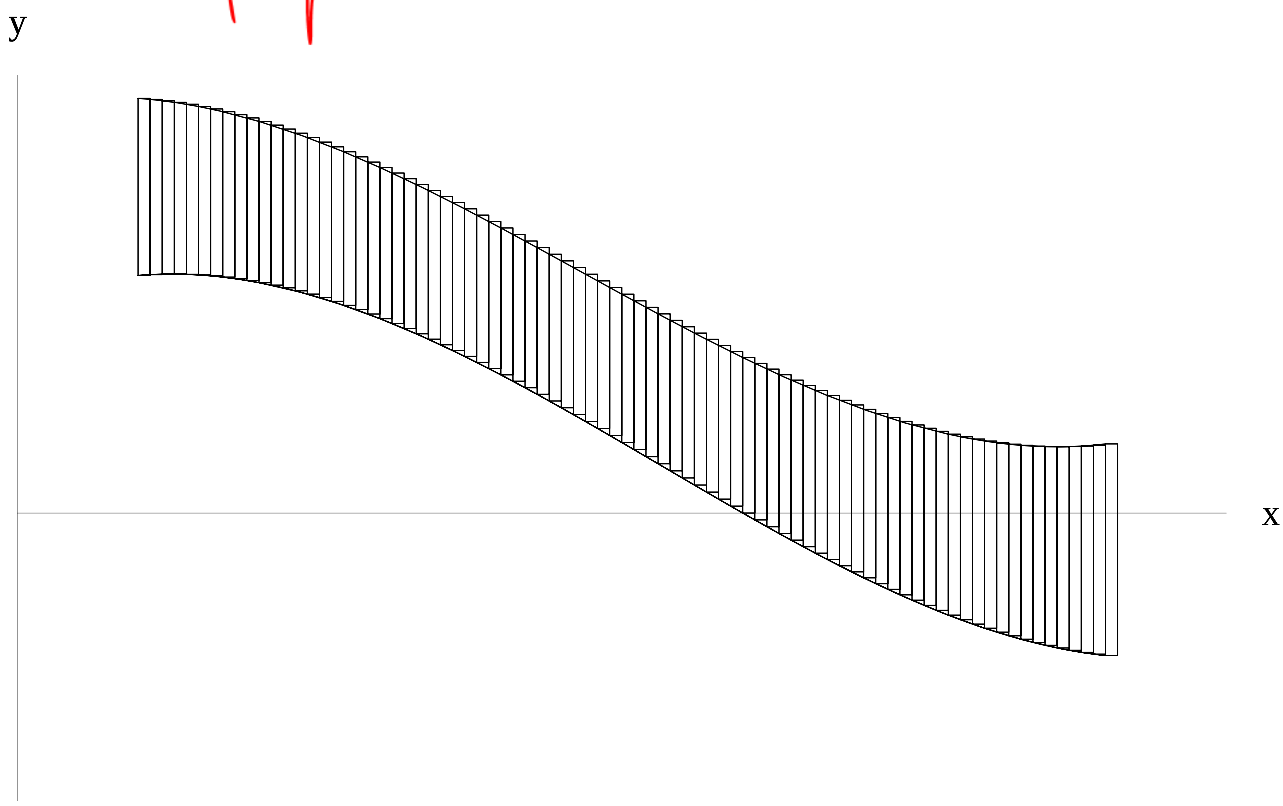
[*Stewart §6.1, §3.4.2 of Notes*]

Let us suppose that we wish to find the area between the two curves $y = f(x)$ and $y = g(x)$ between the ordinates $x = a$ and $x = b$.



The area consists of thin rectangles of area $(g(x) - f(x))\ dx$.

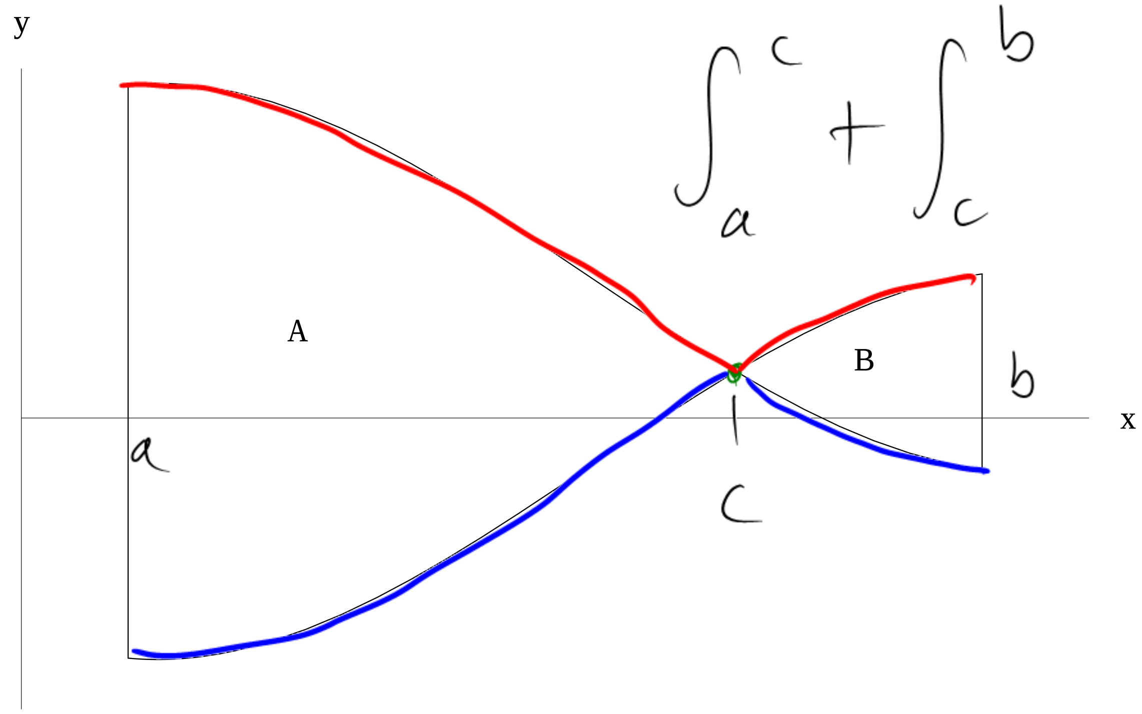
top − bottom



Adding the thin rectangles, we get

$$\text{Area between} = \int_a^b (g(x) - f(x))\, dx$$

It is always best to draw a graph when finding area between curves.



Here you need to split the total integral in two pieces: take the positive integral for A, but the negative of the integral for B.

"Algebraic Area"

$$= \int_a^b (g(x) - f(x)) \ dx$$

$$= A - B$$

"Absolute Area"

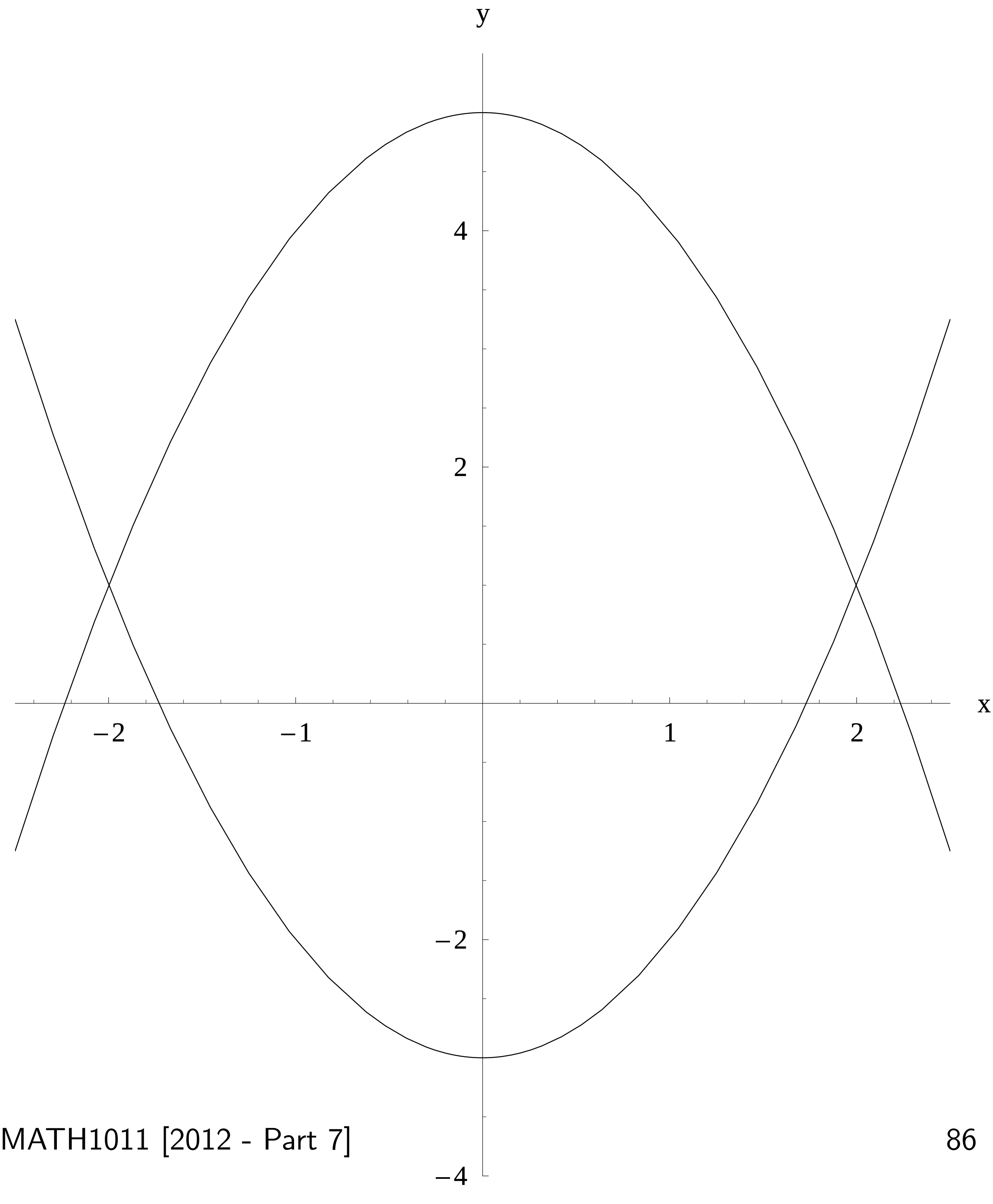
$$= \int_a^b |g(x) - f(x)| \ dx$$

$$= A + B$$

Example

Find the area enclosed by the parabolas:

$$y = x^2 - 3 \text{ and } y = 5 - x^2$$



We need the points of intersection.

$$x^2 - 3 = 5 - x^2$$

$$2x^2 = 8$$

$$x = \pm 2.$$

Area

$$= \int_{-2}^{2} \big((5 - x^2) - (x^2 - 3)\big)\ dx$$

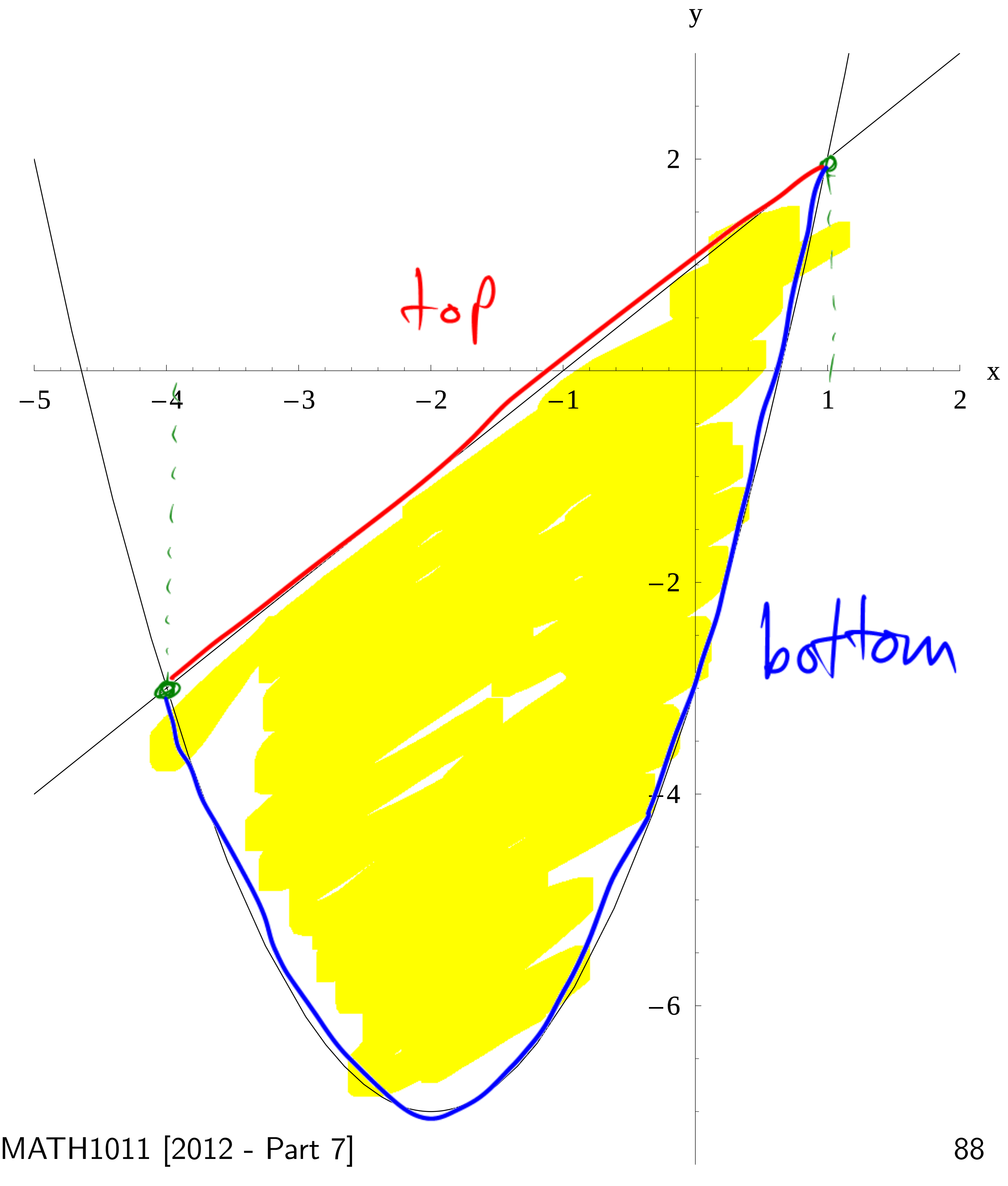
$$= \int_{-2}^{2} \big(8 - 2x^2\big)\ dx$$

$$= \left[8x - \tfrac{2}{3}x^3\right]_{-2}^{2} = 21\tfrac{1}{3}$$

Example

Find the area enclosed by the straight line and parabola:

$$y = x + 1 \text{ and } y = x^2 + 4x - 3$$



We need the points of intersection.

$$x^2 + 4x - 3 = x + 1$$

$$x^2 + 3x - 4 = 0$$

$$(x - 1)(x + 4) = 0$$

$$x = -4, 1.$$

Area

$$= \int_{-4}^{1} \big((x+1) - (x^2 + 4x - 3)\big)\ dx$$

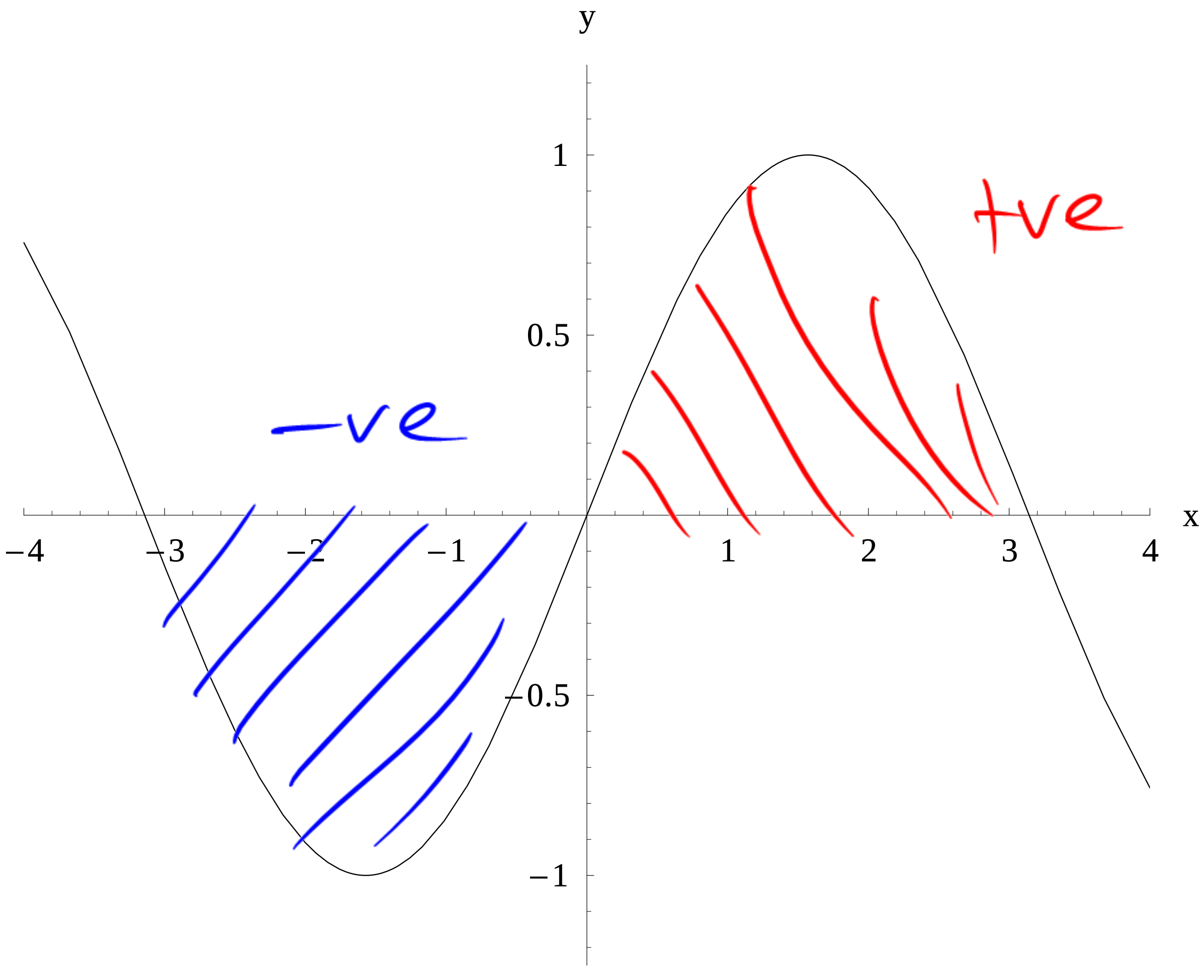
$$= \int_{-4}^{1} \big(-x^2 - 3x + 4\big)\ dx$$

$$= \big[-\tfrac{1}{3}x^3 - \tfrac{3}{2}x^2 + 4x\big]_{-4}^{1}$$

$$= 20\tfrac{5}{6}$$

Example

Find the area enclosed by the curve $y = \sin x$ and the x-axis for $-\pi \leq x \leq \pi$.



The “algebraic” area is clearly 0.

The “absolute area” is[12]

Absolute area

$$= \int_{-\pi}^{0} (0 - \sin x)\ dx + \int_{0}^{\pi} (\sin x - 0)\ dx$$

$$= - \int_{-\pi}^{0} \sin x \, dx + \int_{0}^{\pi} \sin x \, dx$$

$$= - \left[- \cos x \right]_{-\pi}^{0} + \left[- \cos x \right]_{0}^{\pi}$$

$$= 4.$$

[12]Or alternatively you can compute this integral by exploiting symmetry: $2 \int_0^{\pi} \sin x \, dx$.

Improper Integrals

[*Stewart §8.8, Notes §3.5*]

In the last topic we introduced the concept of integration to find the area under the curve $y = f(x)$ for $a \leq x \leq b$, where a, b and $f(a), f(b)$ (and all values in-between) are finite and well defined. They were called *proper integrals*.

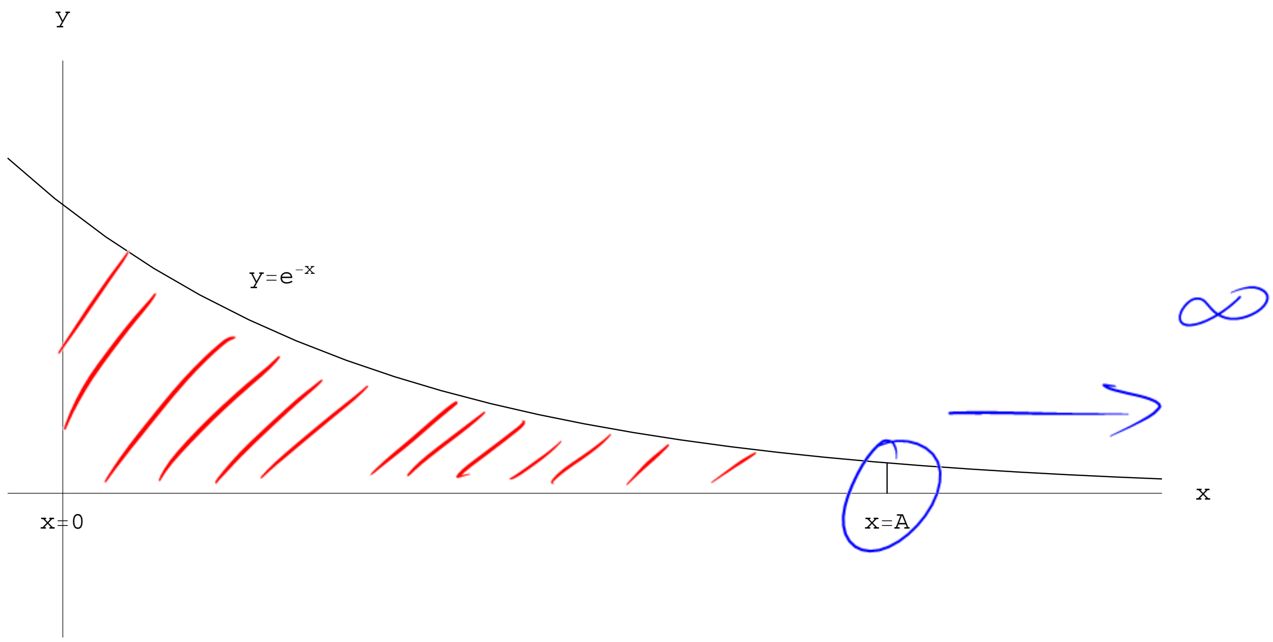
We are now going to look at the interesting cases when these quantities can be *infinite or undefined*. These types of integrals are called *improper integrals*.

Type 1: infinite domains

We will first consider the case where either a or b (or both) in the integration domain are infinite. Let's start with a familiar example.

Example

Consider the following situation:



Area under the curve from 0 to A is

$$\begin{aligned}\int_0^A e^{-x}\,dx &= -\int_0^A -e^{-x}\,dx\\ &= -\left[e^{-x}\right]_0^A\\ &= -e^{-A} + e^0\\ &= 1 - e^{-A}.\end{aligned}$$

As A gets bigger e^{-A} gets smaller and the area under the curve from 0 to A approaches 1; or as $A \to \infty$ the area under the curve from 0 to A approaches 1 (if you're skeptical, put a really big number for A into this formula using your calculator; you should get a really small number).

So the value “1” is a good candidate for the area under the curve $y = e^{-x}$ from 0 to ∞ and it seems reasonable to write

$$\int_0^\infty e^{-x}\,dx := \lim_{A\to\infty}\int_0^A e^{-x}\,dx = 1.$$

We calculate this as

$$\begin{aligned}
\int_0^\infty e^{-x}\,dx &= \lim_{A\to\infty}\int_0^A e^{-x}\,dx \\
&= -\lim_{A\to\infty}\int_0^A -e^{-x}\,dx \\
&= -\lim_{A\to\infty}\left[e^{-x}\right]_0^A \\
&= -\lim_{A\to\infty}\left(e^{-A}-e^0\right) \\
&= \lim_{A\to\infty}\left(1-e^{-A}\right) \\
&= 1.
\end{aligned}$$

But sometimes the limit doesn't exist:

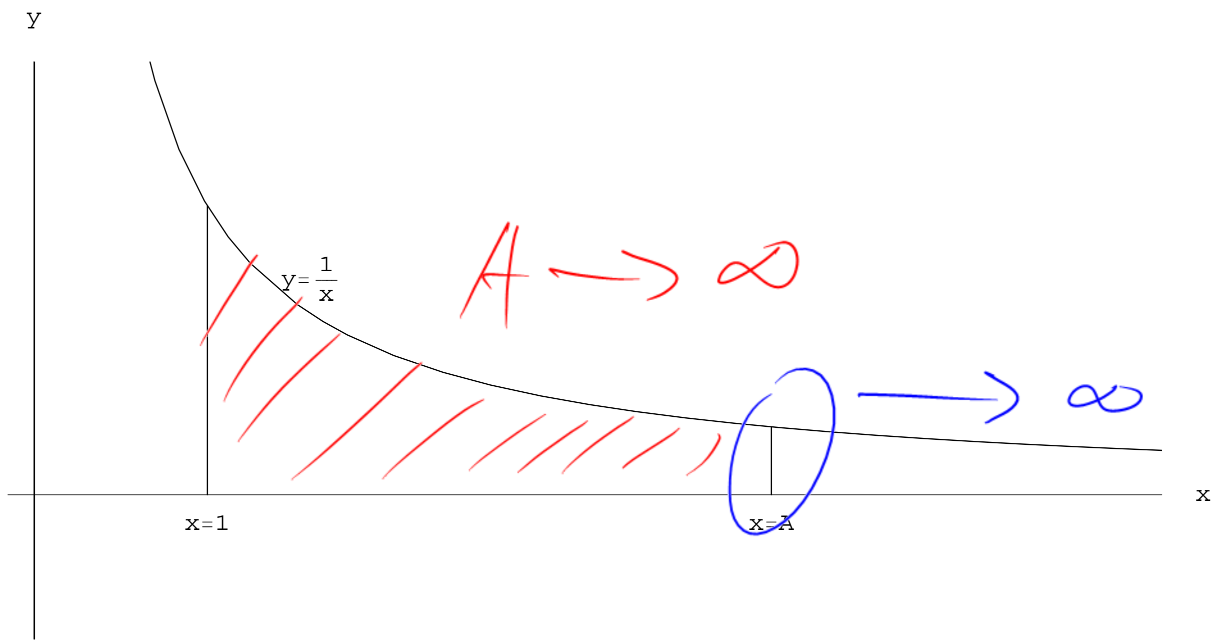
$$\int_1^\infty \frac{dx}{x} = \lim_{A\to\infty} \int_1^A \frac{dx}{x}$$

$$= \lim_{A\to\infty} \left[\ln|x|\right]_1^A$$

$$= \lim_{A\to\infty} (\ln(A) - \ln(1))$$

$$= \lim_{A\to\infty} \ln(A)$$

and this limit does not exist because it does not yield a finite number. We say $\int_1^\infty \frac{dx}{x}$ "does not exist" or "does not converge."



In this case we **colloquially** say

$$\int_1^\infty \frac{dx}{x} = \infty$$

or the area under the curve is infinite, but it is best to stay clear of this language because of the weird things that can happen.

Definition

$$\int_a^\infty f(x)\,dx = L \qquad \text{or converges to } L$$

means for each $A \geq a$

$$\int_a^A f(x)\,dx$$

exists and

$$\lim_{A\to\infty} \int_a^A f(x)\,dx = L.$$

Definition

$\int_a^\infty f(x)\,dx$ does not exist or diverges

means either for some $A \geq a$

$$\int_a^A f(x)\,dx$$

does not exist or the integral exists for all A but

$$\lim_{A\to\infty} \int_a^A f(x)\,dx$$

does not exist.

Example

Evaluate

$$\int_1^\infty \frac{dx}{x^2}.$$

$$\begin{aligned}
\int_1^A \frac{dx}{x^2} &= \int_1^A x^{-2}\, dx \\
&= \left[-x^{-1}\right]_1^A \\
&= -\frac{1}{A} + \frac{1}{1} \\
&= 1 - \frac{1}{A} \\
&\to 1 \quad \text{as} \quad A \to \infty.
\end{aligned}$$

So

$$\int_1^\infty \frac{dx}{x^2} = 1.$$

Definition

$$\int_{-\infty}^{a} f(x)\,dx = L \qquad \text{or converges to } L$$

means for each $A \le a$

$$\int_{A}^{a} f(x)\,dx$$

exists and

$$\lim_{A \to -\infty} \int_{A}^{a} f(x)\,dx = L.$$

Definition

$$\int_{-\infty}^{\infty} f(x)\,dx = L \qquad \text{or converges to } L$$

means, for all finite a,

$$\int_{-\infty}^{a} f(x)\,dx \quad \text{and} \quad \int_{a}^{\infty} f(x)\,dx$$

exist and

$$\int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx = L.$$

(**alternative definition**)

$$\int_{-\infty}^{\infty} f(x)\,dx = L \qquad \text{or converges to } L$$

means, for all $A \geq 0$

$$\int_{-A}^{A} f(x)\,dx$$

exists and

$$\lim_{A\to\infty} \int_{-A}^{A} f(x)\,dx = L.$$

Example

Investigate $\displaystyle\int_{-\infty}^{0} \frac{dx}{(5-x)^3}$.

$$\int_A^0 \frac{dx}{(5-x)^3} = -\int_A^0 -(5-x)^{-3}\,dx$$

$$= -\left[\frac{(5-x)^{-2}}{-2}\right]_A^0$$

$$= \frac{1}{50} - \frac{1}{2(5-A)^2}$$

$$\to \frac{1}{50} \quad \text{as} \quad A \to -\infty$$

$$\therefore \int_{-\infty}^{0} \frac{dx}{(5-x)^3} = \frac{1}{50}.$$

Example

Investigate $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\,dx$.

$$
\begin{aligned}
\int_A^0 xe^{-x^2}\,dx &= -\frac{1}{2}\int_A^0 -2xe^{-x^2}\,dx \\
&= -\tfrac{1}{2}\left[e^{-x^2}\right]_A^0 \\
&= -\tfrac{1}{2}\left(e^{0^2} - e^{A^2}\right) \\
&= \tfrac{1}{2}e^{-A^2} - \tfrac{1}{2} \\
&\to -\tfrac{1}{2} \quad \text{as} \quad A \to -\infty
\end{aligned}
$$

$$\therefore \int_{-\infty}^{0} xe^{-x^2}\,dx = -\tfrac{1}{2}.$$

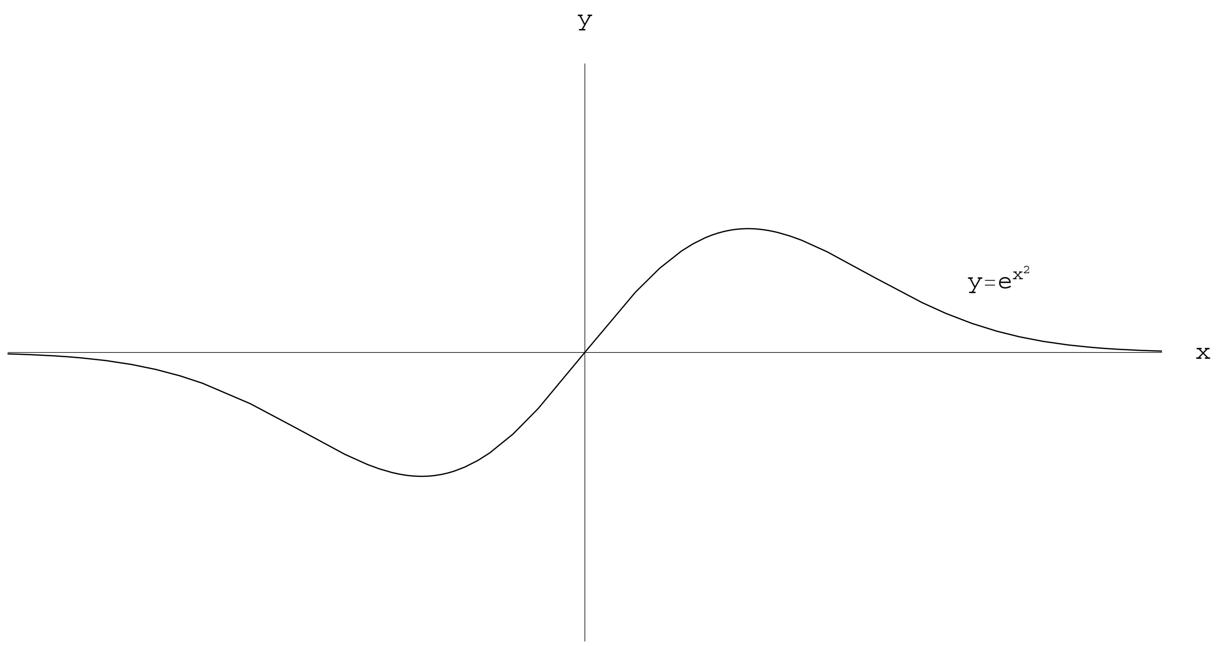
$$\begin{aligned}
\int_0^A xe^{-x^2}\,dx &= -\frac{1}{2}\int_0^A -2xe^{-x^2}\,dx \\
&= -\tfrac{1}{2}\left[e^{-x^2}\right]_0^A \\
&= -\tfrac{1}{2}\left(e^{-A^2} - e^{0^2}\right) \\
&= \tfrac{1}{2} - \tfrac{1}{2}e^{-A^2} \\
&\to \tfrac{1}{2} \quad \text{as} \quad A \to \infty
\end{aligned}$$

$$\therefore \int_0^\infty xe^{-x^2}\,dx = \tfrac{1}{2}.$$

It follows that

$$\int_{-\infty}^{\infty} xe^{-x^2}\, dx = 0$$

which can be seen from the graph below:



Alternatively, we can do the limits in one step! With infinity limits on both ends, we make a substitution A for both the upper and lower limits, and then take the limit at the end for both parts at once, hence eliminating the need to split up the integral into two parts.

BUT beware! This may not work if you do not have EACH individual piece of the integral existing/converging!

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx$$

$$= \lim_{A\to\infty} \int_{-A}^{A} xe^{-x^2}\,dx$$

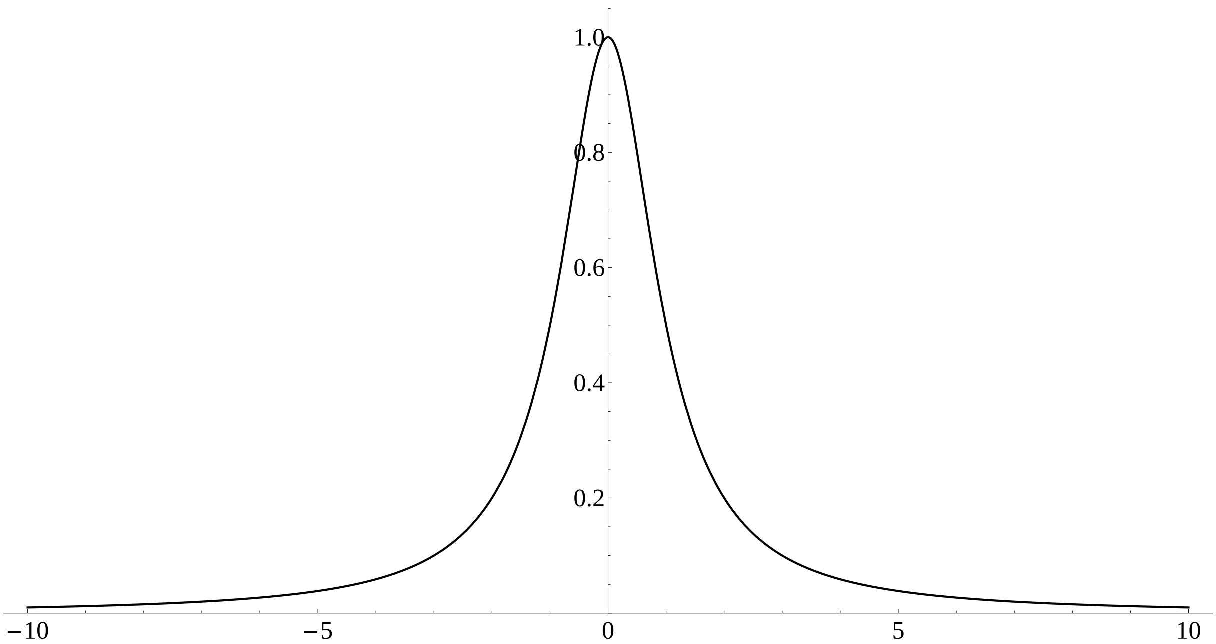
$$= \lim_{A\to\infty} \left\{ -\frac{1}{2} \int_{-A}^{A} -2xe^{-x^2}\,dx \right\}$$

$$= -\frac{1}{2} \lim_{A\to\infty} \left[e^{-x^2} \right]_{-A}^{A}$$

$$= -\frac{1}{2} \lim_{A\to\infty} \left(e^{-A^2} - e^{-(-A)^2} \right)$$

$$= -\frac{1}{2} \lim_{A\to\infty} 0 = 0.$$

I now want to show you my favourite formula for π. Consider the graph $y = \frac{1}{1+x^2}$. It's a simple "pulse" shaped graph:



From highschool you may have learnt how to evaluate $\int \frac{dx}{1+x^2}$. If so, then consider finding the area the entire graph:

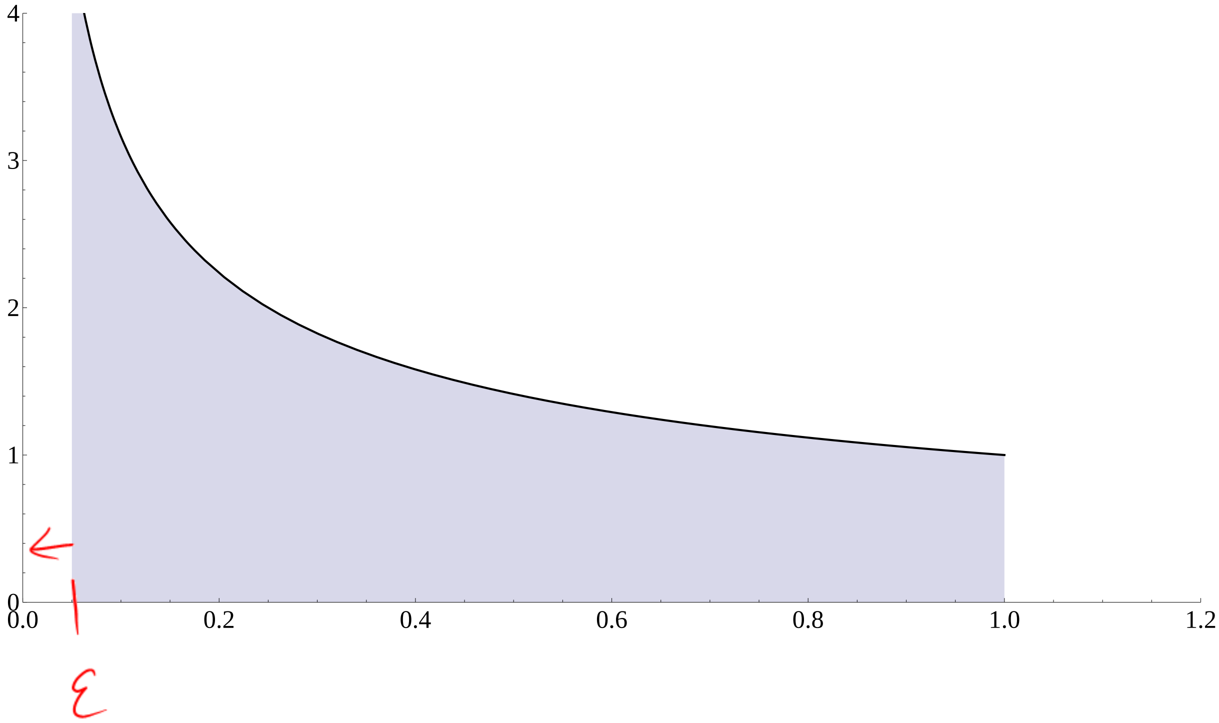
$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dx}{1+x^2} &= \lim_{A\to\infty} \int_{-A}^{A} \frac{dx}{1+x^2} \\
&= 2 \lim_{A\to\infty} \int_{0}^{A} \frac{dx}{1+x^2} \\
&= 2 \lim_{A\to\infty} \left[\tan^{-1} x\right]_0^A \\
&= 2 \lim_{A\to\infty} \left[\tan^{-1} A - \tan^{-1} 0\right] \\
&= 2 \times \left[\frac{\pi}{2} - 0\right] = \pi.
\end{aligned}$$

Type 2: discontinuities

We will now consider the case where the integrand is discontinuous at some isolated points in the integration domain. Again we shall consider the following example

$$\int_0^1 \frac{dx}{x^{\frac{1}{2}}}.$$

Why isn't this a common proper integral? Because $\frac{1}{x^{\frac{1}{2}}}$ is **undefined** when $x = 0$. So instead we start at a small ϵ away from the singularity.



4
3
2
1
0
0.0
0.2
0.4
0.6
0.8
1.0
1.2
ε

We give it meaning as follows:

$$\begin{aligned}\lim_{\epsilon\to 0}\int_\epsilon^1 \frac{dx}{x^{\frac{1}{2}}} &= \lim_{\epsilon\to 0}\int_\epsilon^1 x^{-\frac{1}{2}}\,dx \\ &= \lim_{\epsilon\to 0}\left[2x^{\frac{1}{2}}\right]_\epsilon^1 \\ &= \lim_{\epsilon\to 0}\left[2-2\epsilon^{\frac{1}{2}}\right] \\ &= 2.\end{aligned}$$

We say that $\displaystyle\int_0^1 \frac{dx}{x^{\frac{1}{2}}}$ exists or converges and write

$$\int_0^1 \frac{dx}{x^{\frac{1}{2}}} := \lim_{\epsilon\to 0}\int_\epsilon^1 \frac{dx}{x^{\frac{1}{2}}} = 2.$$

In general, when the integrand is discontinuous at isolated points we can cut away the discontinuities and take limits in the above way to gather good definition of the integral. We then take the limit as the gap approaches zero.

Example

Investigate

$$\int_2^4 \frac{dx}{(x-3)^3}.$$

The integrand, $\dfrac{1}{(x-3)^3}$ is continuous everywhere except at $x = 3$ where it is discontinuous.

First we will consider the section where $2 \leq x < 3$.

$$\int_2^{3-\epsilon} \frac{dx}{(x-3)^3} = \int_2^{3-\epsilon} (x-3)^{-3}\,dx$$

$$= \left[-\tfrac{1}{2}(x-3)^{-2}\right]_2^{3-\epsilon}$$

$$= -\frac{1}{2\epsilon^2} + \frac{1}{2}$$

This does not have a limit as $\epsilon \to 0$.

So $\displaystyle\int_2^4 \frac{dx}{(x-3)^3}$ does not exist (or does not converge).

Example

Investigate

$$\int_1^6 \frac{dx}{\sqrt{x-1}}.$$

$\frac{1}{0}$ undefined

The integrand, $\dfrac{1}{\sqrt{x-1}}$ is continuous everywhere except $x = 1$, where it is discontinuous. Let's start integrating from a little bit past the lower limit, namely $1 + \epsilon$.

$$\int_1^6 \frac{dx}{\sqrt{x-1}} = \lim_{\epsilon \to 0} \int_{1+\epsilon}^6 (x-1)^{-\frac{1}{2}}\, dx$$

$$= \lim_{\epsilon \to 0} \left[2(x-1)^{\frac{1}{2}} \right]_{1+\epsilon}^6$$

$$= 2 \lim_{\epsilon \to 0} \left[\sqrt{5} - 2\epsilon^{\frac{1}{2}} \right]$$

$$= 2\sqrt{5}$$

So

$$\int_1^6 \frac{dx}{\sqrt{x-1}} = 2\sqrt{5}.$$

Example

Investigate

$$\int_0^1 \frac{x\,dx}{\sqrt[3]{1-x^2}}.$$

The integrand, $\dfrac{x}{\sqrt[3]{1-x^2}}$ is continuous everywhere except $x = \pm 1$ where it is discontinuous.

For this integral, let's integrate up to just a little bit less than the upper limit, namely $1 - \epsilon$.

$$\int_0^{1-\epsilon} \frac{x\,dx}{\sqrt[3]{1-x^2}}$$

$$= -\frac{1}{2}\int_0^{1-\epsilon} (-2x)(1-x^2)^{-\frac{1}{3}}\,dx$$

$$= -\tfrac{1}{2}\left[\tfrac{3}{2}(1-x^2)^{\frac{2}{3}}\right]_0^{1-\epsilon}$$

$$= \tfrac{3}{4} - \tfrac{3}{4}(2\epsilon - \epsilon^2)^{\frac{2}{3}}$$

$$\to \tfrac{3}{4} \quad \text{as} \quad \epsilon \to 0$$

So

$$\int_0^1 \frac{x\,dx}{\sqrt[3]{1-x^2}} = \tfrac{3}{4}.$$

Example[13]

Investigate

$$\int_{-1}^{1} \frac{dx}{x^2}.$$

The integrand, $\frac{1}{x^2}$ is continuous everywhere except at $x = 0$ where it is discontinuous (in fact is not defined at 0).

Let's start by integrating one side first, where $-1 \leq x < 0$.

[13] This is a very common MATH1003 question!

$$\int_{-1}^{-\epsilon} \frac{dx}{x^2} = \int_{-1}^{-\epsilon} x^{-2}\, dx$$

$$= \left[-x^{-1}\right]_{-1}^{-\epsilon}$$

$$= \frac{1}{\epsilon} - 1$$

This does not have a limit as $\epsilon \to 0$.

So $\displaystyle\int_{-1}^{1} \frac{dx}{x^2}$ does not exist (or does not converge).

What if we did the integral naïvely?

$$\int_{-1}^{1} \frac{dx}{x^2} = \int_{-1}^{1} x^{-2}\, dx$$

$$= \left[-x^{-1}\right]_{-1}^{1} = -2.$$

So it seems that the area is well defined and is negative. But look at the graph! The area is positive! This contradiction occurs because we attempted to integrate over a singularity, so the integral is invalid.