Applications of Calculus MATH1011 – 2012 Summer

Lecture 8: Sequences and Series

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Lecture Outline

- Sequences and Series
- Sigma Notation
- Useful concepts: Linearity of Summation and Shifting Indices
- Closed-form Formulas
- Collapsing Sums
- Arithmetric Progressions (AP)
- Geometric Progressions (GP)
- Application to Financial Problems
- Limiting Sum of a GP

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Sequences and Series (cont.)

A **sequence** is an ordered set of numbers. Each number is called a **term** of the sequence. Notation-wise, T_k denotes the k-th term of the sequence.

A **series** is a sum of the terms of a sequence. Notation-wise, S_n denotes the sum of the first n terms of the sequence.

Example

Consider the sequence $2, 4, 6, 8, 10, 12, \ldots$

The fourth term in the sequence T_4 is 8.

The sum of the first 5 terms of the sequence S_5 is $2 + 4 + 6 + 8 + 10 = 30$.

Sequences and Series (cont.)

Example

Consider the sequence defined by $T_k = k$. Then:

$T_1 = 1$
$T_2 = 2$
$T_3 = 3$
$T_4 = 4$
etc.

So the sequence (of counting numbers) written as a list is:
$1, 2, 3, 4, 5, 6, \ldots$.

S_{10} for instance, equals
$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$.

Sequences and Series (cont.)

Example

Consider the sequence defined by $T_k = k^2$. Then:

$T_1 = 1^2 = 1$
$T_2 = 2^2 = 4$
$T_3 = 3^2 = 9$
$T_4 = 4^2 = 16$
etc.

So the sequence (of perfect squares) written as a list is:
$1, 4, 9, 16, 25, 36, 49, 64, 81, \ldots$.

S_6 for instance, equals $1 + 4 + 9 + 16 + 25 + 36 = 91$.

What if we wanted a big sum, like S_{100}? For our counting number sequence, that would be written as $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + \cdots + 99 + 100$.

To condense the way we write sums, there is something called the **Sigma notation** for summation:

$$\sum_{k=m}^{n} T_k$$

The above says to sum up all the terms from the m-th term T_m all the way up to the n-th term T_n.

Sigma Notation (cont.)

Example

So S_{100} in Sigma notation can be written as:

$$S_{100} = \sum_{k=1}^{100} T_k = \sum_{k=1}^{100} k.$$

The bottom index $k = 1$ indicates we are starting our sum from the 1st term $T_1 = 1$, and the upper index of 100 indicates we keep summing until we get to the 100-th term $T_{100} = 100$.

Sigma Notation (cont.)

To expand a Sigma expression:

"*for each number k from m (bottom index) to n (top index), you write down the term* T_k *and then add them all up.*"

Example

Expand

$$\sum_{k=1}^{6} k^2.$$

So for each number from k from 1 to 6, we write down $T_k = k^2$ and add everything up. Hence we get 1 + 4 + 9 + 16 + 25 + 36 = 91. Note: this is S_6 for our perfect squares sequence.

Sigma Notation (cont.)

Example

Consider a sequence $\frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \ldots$ defined by:

$$T_k = \frac{k}{k+2}.$$

Then the sum of the first 20 terms $S_{20} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \cdots + \frac{20}{22}$ can be written as the following in Sigma notation:

$$\sum_{k=1}^{20} \frac{k}{k+2}$$

Linearity of Summation

Example

Consider the following series:

$$\sum_{k=0}^{4}(3k^2 + 5k - 4)$$

$$= (3 \times 0^2 + 5 \times 0 - 4) + (3 \times 1^2 + 5 \times 1 - 4) + (3 \times 2^2 + 5 \times 2 - 4)$$
$$+(3 \times 3^2 + 5 \times 3 - 4) + (3 \times 4^2 + 5 \times 4 - 4)$$

$$= -4 + 4 + 18 + 38 + 64 = 120$$

Linearity of Summation (cont.)

But we can also evaluate this by 'splitting up the sum':

$$\sum_{k=0}^{4}(3k^2+5k-4) = 3\sum_{k=0}^{4}k^2 + 5\sum_{k=0}^{4}k - 4\sum_{k=0}^{4}1$$

$= 3(0^2+1^2+2^2+3^2+4^2)+5(0+1+2+3+4)-4(1+1+1+1+1)$
$= 3\times 30+5\times 10-4\times 5 = 120$ as expected.

This 'splitting up the sum' concept is formally known as the **linearity of summation** principle:

$$\sum_{k}[af(k)+bg(k)] = a\sum_{k}f(k) + b\sum_{k}g(k)$$

Shifting Indices

It's important to realise that the same summation can be written in multiple ways under Sigma notation by "shifting the indices".

Example

The sum 1 + 2 + 3 + 4 + 5 + 6 for the sequence $T_k = k$ can be written as:

$$\sum_{k=1}^{6} k \quad \text{or} \quad \sum_{k=0}^{5} (k+1).$$

More generally:

$$\sum_{k=m}^{n} T_k = \sum_{k=m-1}^{n-1} T_{k+1}.$$

Closed-form Formulas

So far, we've looked at series (which are basically just sums) and how to write them compactly using Sigma notation. But we also need to **work them out** (preferably without a calculator).

E.g. S_{100} for our counting number sequence is

$$1 + 2 + 3 + 4 + \cdots + 100 = \sum_{k=1}^{100} k = ?$$

Is there a "closed-form" formula we can derive for S_n for any n?

Closed-form Formulas (cont.)

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{1}{2} n(n+1).$$

So $S_{100} = \sum_{k=1}^{100} k = 1 + 2 + \cdots + 100 = \frac{1}{2} 100(100+1) = 5050.$

The 'proof' of this result comes from this clever observation.

$2 \sum_{k=1}^{n} k = (1+2+3+\cdots+n)+(n+(n-1)+(n-2)+\cdots+1) = n(n+1)$

For instance, $(1+2+3+4+5+6)+(6+5+4+3+2+1)$
$= 1+2+3+4+5+6+6+(6-1)+(6-2)+(6-3)+(6-4)+(6-5)$
$= 6+6+6+6+6+6+6$
$= 6 \times 7.$

Closed-form Formulas (cont.)

What about our perfect squares sequence $T_k = k^2$?

$$\sum_{k=1}^{n} k^2 = 1 + 4 + 9 + 16 + 25 + \cdots + n^2 = \frac{1}{6}(n+1)(2n+1).$$

See George's lecture notes for a rigorous proof of this result!

Closed-form Formulas (cont.)

Finally, it so happens that

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2 = \frac{1}{4}n^2(n+1)^2$$

That is, the sum of the first n cubed terms amazingly turns out to be equal to the square of the sum of the first n natural numbers! Again, refer to George's notes for a formal proof.

Collapsing Sums

Recall that the closed-form formula

$$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$$

was obtained courtesy of a ‘bunch of things cancelling out’ in the expansion of $2\sum_{k=1}^{n} k$.

Such a characteristic forms the essence of a particular type of series known as **collapsible sums**:

$$\sum_{k=m}^{n} (f_{k+1} - f_k).$$

for some sequence f_k.

Collapsing Sums (cont.)

To see this cancelling phenomenon, let's expand the sum:

$\sum_{k=m}^{n}(f_{k+1} - f_k)$

$= (f_{m+1} - f_m) + (f_{m+2} - f_{m+1}) + (f_{m+3} - f_{m+2}) + \cdots + (f_{n+1} - f_n)$

$= f_{n+1} - f_m$

Example

Let $f_k = k^3$. Then:

$$\sum_{k=1}^{6}[f_{k+1} - f_k] = \sum_{k=1}^{6}[(k+1)^3 - k^3]$$

$(2^3 - 1^3) + (3^3 - 2^3) + (4^3 - 3^3) + (5^3 - 4^3) + (6^3 - 5^3) + (7^3 - 6^3)$
$= 7^3 - 1^3 = 342.$

Arithmetic Progressions (AP)

Consider a sequence:

$$3, 7, 11, 15, 19, \ldots.$$

Each term is formed by *adding a common number* to the preceding term. Here that number is 4.

The general term T_k can be written as 3 + (k - 1)4.

T_n of a AP

Here's another example: $5, 7, 9, 11, 13, 15, \ldots$.

The common difference here is 2 and the general term T_k can be written as 5 + (k - 1)2.

These sequences are examples of **Arithmetic Progressions** – sequences in which each term after the ***first term*** *a* is formed by adding a ***common 'difference'*** *d* to the preceding term.

In particular:

The k-th term of an arithmetic progression $a, a+d, a+2d, a+3d, \ldots$ is given by

$$T_k = a + (k-1)d.$$

T_n of a AP (cont.)

Example

Find the 20th term of the sequence $5, 8, 11, 14, 17, 20, \ldots$.

Solution

The sequence is a AP where $a = 5$ and $d = 3$, so
$T_k = 5 + (k - 1)3$.

Now $T_{20} = 5 + (20 - 1)3 = 62$, hence the 20th term is 62.

T_n of a AP (cont.)

Example

Which term of the sequence $15, 11, 7, \ldots$ is -33?

Solution

The sequence is a AP where $a = 15$ and $d = -4$ so $T_k = 15 + (k-1)(-4)$.

Setting T_k to be -33, we have the equation:

$$-33 = 15 + (k-1)(-4),$$

hence $k = 13$.

-33 is therefore the 13th term.

S_n of a AP

Is there a closed-form formula for the **sum** of the first n terms of a general AP:

$$S_n = \sum_{k=1}^{n} T_k = a + (a+d) + (a+2d) + \cdots + (a+(n-1)d)?$$

Yes.

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right].$$

S_n of a AP (cont.)

Here's a proof of this result:

$$a + (a + d) + (a + 2d) + \cdots + (a + (n-1)d)$$

$$= \sum_{k=1}^{n} (a + (k-1)d)$$

$$= a \sum_{k=1}^{n} 1 + d \sum_{k=1}^{n} k - d \sum_{k=1}^{n} 1$$

$$= an + d\tfrac{1}{2}n(n+1) - dn$$

$$= \tfrac{1}{2}n\,(2a + (n-1)d)$$

Moreover, note that this is equal to $n \times \frac{1}{2}(a + (a + (n-1)d))$
$=$ number of terms $\times$ average of first & last terms.

Example

Find the sum of the first 20 terms of the series
$7 + 12 + 17 + 22 + \dots$

Solution

The sequence $7, 12, 17, 22, \dots$ is a AP with $a = 7$ and $d = 5$.

Therefore

$S_{20} = \frac{n}{2}\left[2a + (n-1)d\right]$ where $n = 20, a = 7$ and $d = 5$.

$= \frac{20}{2}\left[2(7) + (20-1)5\right]$

$= 1090.$

S_n of a AP (cont.)

Example

How many terms in the series $48 + 44 + 40 + \ldots$ need to be taken to give a sum of 308?

Solution

Using

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right], \quad \text{where } a = 48, d = -4 \text{ and } S_n = 308$$

we have

$$308 = \frac{n}{2}[96 + (n-1)(-4)],$$

$$\text{i.e.} \quad n^2 - 25n + 154 = (n-11)(n-14) = 0.$$

Hence $n = 11$ or 14. So the sum of 11 terms = sum of 14 terms = 308. (What does this tell you about $T_{12} + T_{13} + T_{14}$)?

Geometric Progressions (GP)

Consider a sequence:

$$2, 6, 18, 54, 162, \ldots.$$

Each term is formed by *multiplying a constant number* to the preceding term. Here that constant is 3.

The general term T_k can be written as $2(3)^{k-1}$.

T_n of a GP

Here's another example: $32, 16, 8, 4, 2, \ldots$.

The common constant here is $\frac{1}{2}$ and the general term T_k can be written as $32(\frac{1}{2})^{k-1}$.

These sequences are examples of **Geometric Progressions** – sequences in which each term after the ***first term*** a is obtained from the preceding one by multiplying it by a ***common 'ratio'*** r.

In particular:

The k-th term of a geometric progression $a, ar, ar^2, ar^3, \ldots$ is given by

$$T_k = ar^{k-1}.$$

T_n of a GP (cont.)

Example

Find the 7th term of the sequence $54, -18, 6, \ldots$.

Solution

The sequence is a GP where $a = 54$ and $r = -\frac{1}{3}$, so $T_k = 54(-\frac{1}{3})^{k-1}$.

Now $T_7 = 54(-\frac{1}{3})^{7-1} = \frac{2}{27}$, hence the 7th term is $\frac{2}{27}$.

S_n of a GP

Is there a closed-form formula for the **sum** of the first n terms of a general GP:

$$S_n = \sum_{k=1}^{n} T_k = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}?$$

Yes.

$$S_n = \frac{a\left(1 - r^n\right)}{1 - r}.$$

S_n of a GP (cont.)

Here's a proof of this result:

$\sum_{k=1}^{n} ar^{k-1} = a + ar + ar^2 + \ldots + ar^{n-1}$

$r\left(\sum_{k=1}^{n} ar^{k-1}\right) = ar + ar^2 + ar^3 + \ldots + ar^n$

Subtracting the second equation from the first, we get

$(1-r)\left(\sum_{k=1}^{n} ar^{k-1}\right) = a(1-r^n).$

Dividing both sides by $1-r$, we thus get

$\sum_{k=1}^{n} ar^{k-1} = \dfrac{a(1-r^n)}{1-r} = \dfrac{a(r^n-1)}{r-1}.$

S_n of a GP (cont.)

Example

Find the sum of the first 9 terms of the series $32 + 16 + 8 + \dots$

Solution

The sequence $32, 16, 8, \dots$ is a GP with $a = 32$ and $r = \frac{1}{2}$.

Therefore

$$S_9 = \frac{a(1 - r^n)}{1 - r} \text{ where } n = 9, a = 32 \text{ and } r = \tfrac{1}{2}.$$

$$= \frac{32(1 - (\frac{1}{2})^9}{1 - \frac{1}{2}}$$

$$= 59.88.$$

Example

\$1000 is invested at the beginning of 1994 and \$50 is added to the investment at the beginning of each year for the next 19 years. Interest is paid at 8% per annum in December of each year. What is the total investment at the end of 20 years?

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Applications to Financial Problems (ct.)

Solution

Beginning of 1st year
1000

End of 1st year (after 1st year's interest compounded)
$1000(1.08)$

Beginning of 2nd year (after \$50 *increment)*
$1000(1.08) + 50$

End of 2nd year (after 2nd year's interest compounded)
$(1000(1.08) + 50)(1.08)$
$= 1000(1.08)^2 + 50(1.08)$

Applications to Financial Problems (ct.)

Beginning of 3rd year (after \$50 *increment)*
$1000(1.08)^2 + 50(1.08) + 50$

End of 3rd year (after 3rd year's interest compounded)
$1000(1.08)^2 + 50(1.08) + 50)(1.08)$
$= 1000(1.08)^3 + 50(1.08)^2 + 50(1.08)$

Beginning of 4th year (after \$50 *increment)*
$1000(1.08)^3 + 50(1.08)^2 + 50(1.08) + 50$

End of 4th year (after 4th year's interest compounded)
$(1000(1.08)^3 + 50(1.08)^2 + 50(1.08) + 50)(1.08)$
$= 1000(1.08)^4 + 50(1.08)^3 + 50(1.08)^2 + 50(1.08)$

Notice the *pattern*?

Applications to Financial Problems (ct.)

End of 20th year

$$= 1000(1.08)^{20} + 50(1.08)^{19} + 50(1.08)^{18} + \cdots + 50(1.08)$$

$$= 1000(1.08)^{20} + 50(1.08)[1.08^{18} + (1.08)^{17} + \cdots + 1]$$

$$= 1000(1.08)^{20} + 50(1.08)\sum_{k=0}^{18} 1(1.08)^k$$

$$= 1000(1.08)^{20} + 50(1.08)\frac{1 - 1.08^{19}}{1 - 1.08}$$

$$= \$6899.$$

Applications to Financial Problems (ct.)

Example

Moreover, suppose that the investor wishes to have the money paid out in six equal installments: each installment is to be paid at the beginning of the year; payment starts immediately. Assuming that interest continues to be paid at the rate of 8% per annum, find the amount of each installment.

Applications to Financial Problems (ct.)

Solution

Let $\$A$ be the amount paid out at the beginning of each year. (Why isn't $\$A$ simply $\$6899/6 = \$1150$?)

End of 20th year (i.e. situation at the end of the last example)
6899

Beginning of 21st year (after 1st payment)
$6899 - A$

End of 21st year (after 21st year interest payment)
$(6899 - A)(1.08)$
$= 6899(1.08) - A(1.08)$

Beginning of 22nd year (after 2nd payment)
$6899(1.08) - A(1.08) - A$

Applications to Financial Problems (ct.)

End of 22nd year (after 22nd year interest payment)
$(6899(1.08) - A(1.08) - A)(1.08)$
$= 6899(1.08)^2 - A(1.08)^2 - A(1.08)$

Beginning of 23rd year (after 3rd payment)
$6899(1.08)^2 - A(1.08)^2 - A(1.08) - A$

... etc.

Beginning of 26th year (after the 6th and last payment)
$6899(1.08)^5 - A(1.08)^5 - A(1.08)^4 - \cdots - A$

Applications to Financial Problems (ct.)

$$= 6899(1.08)^5 - A\left[(1.08)^5 + (1.08)^4 + \cdots + 1\right]$$

$$= 6899(1.08)^5 - A\sum_{k=1}^{6} 1(1.08)^{k-1}$$

$$= 6899(1.08)^5 - A\,\frac{1 - 1.08^6}{1 - 1.08}$$

which must be 0, since the amount of money left in the account after the last payment must be nil.

Applications to Financial Problems (ct.)

Hence solving for A in the equation

$$6899(1.08)^5 - A\,\frac{1-1.08^6}{1-1.08} = 0,$$

we obtain $A = \$1382$.

In summary, given \$6899 in a bank account with an 8% interest rate – handing out exactly 6 payments over 6 years such that we have no money left in the account after the last payment means each payment should be \$1382.

(Notice how this amount is a bit more than the simplified \$1150 figure, which assumes no interest on the account.)

Applications to Financial Problems (ct.)

Example

Suppose that \$1000 is invested at 10% and that interest is added:

- annually,
- monthly,
- daily,
- hourly,
- continuously.

What is the investment at the end of one year?

Applications to Financial Problems (ct.)

Annually (1 compound period):

$1000\left(1+\frac{0.1}{1}\right)^{1}=\$1100$

Monthly (12 compound periods):

$1000\left(1+\frac{0.1}{12}\right)^{12}=\$1104.70$

Daily (365 compound periods):

$1000\left(1+\frac{0.1}{365}\right)^{365}=\$1105.10$

Hourly (8760 compound periods):

$1000\left(1+\frac{0.1}{8760}\right)^{8760}=\$1105.17$

Continuously ('infinitely many' compound periods):

$\lim_{n\to\infty}1000\left(1+\frac{0.1}{n}\right)^{n}=\$1105.17$

So it doesn't matter how often you compound the interest, there *is* still a **limit** to how much your investment can grow!

An Interesting Fact

Recall from high school that e is a special number between 2 and 3 defined so that the derivative of e^x is equal to itself.

But suppose you invest $1 at 100% interest that is compounded continuously. Then the amount of your investment after one year is actually... the constant e!

$$\lim_{n\to\infty} 1 \times \left(1 + \frac{1}{n}\right)^n = e \approx 2.71828.$$

In fact, this limit is often taken as the *definition* of the number e in higher-level mathematics.

Limiting Sum of a GP

Recall that the sum of the first n terms of a GP $T_k = ar^{k-1}$ is given by:

$$\frac{a\,(1 - r^n)}{1 - r}.$$

However if $|r| < 1$, then $r^n \to 0$ as $n \to \infty$. In other words, the r^n in the formulas *vanishes* when $|r| < 1$ and so we get a neat formula for an infinite sum of a GP when the size of the ratio is less than 1:

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1 - r}$$

Limiting sum of a GP (cont.)

Example

Find $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

Solution

In Sigma notation, this sum is given by

$$\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{k-1}.$$

Here $|r| = |\frac{1}{2}| < 1$ and so

$$\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{k-1} = \frac{1}{1 - \frac{1}{2}} = 2.$$

Limiting sum of a GP (cont.)

Example

Write the recurring decimal $0.123123123\ldots$ as a fraction.

Solution

$$0.123123123\ldots$$

$$= \frac{123}{10^3} + \frac{123}{10^6} + \frac{123}{10^9} + \ldots$$

$$= \frac{123}{10^3}\left[1 + \frac{1}{10^3} + \frac{1}{10^6} + \ldots\right]$$

$$= \frac{123}{10^3}\left[\frac{1}{1 - \frac{1}{10^3}}\right]$$

$$= \frac{123}{1000} \times \frac{1000}{999} = \frac{123}{999}.$$

Special Series

Here are some interesting formulas using infinite series (neither AP nor GP) for π and e:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}.$$

$$\sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots = e.$$

Indeed infinite series are very common in maths and calculus, and we will see some applications in the next topic: Integration!