The University of Sydney
MATH1011 - Applications of Calculus

Summer School **Assignment** 2012

This assignment allows you to write out careful solutions to typical MATH1011 exam questions. The assignment has a total of **90 marks**, and is worth **25%** of the course. You are encouraged to work with your peers, though your assignment solutions**must** be your own work. Plagiarism will NOT be tolerated and sources used to obtain answers (including *Wolfram Alpha*) MUST be cited properly with all your working shown. We are yet to cover all the material that these questions draw from in lectures; this assignment is designed to be worked on over the duration of the course. The assignment is due at **10am on Tuesday the 14th of February (*before* the start of the final lecture)**. Please ask George or the tutors for assistance if you are struggling. Please attach a signed maths assignment cover sheet, available from `http://www.maths.usyd.edu.au/u/UG/asscover.pdf`

1. (i) [**3 marks**] A variable y is oscillating sinusoidally in a 24 hour cycle. It reaches its maximum value of 7 at 9am every morning, and its minimum value of 1 at 9pm every night. Find a formula for y as a function of t (measured in hours), given that it is midnight when $t = 0$. (*Hint: draw a diagram!*)

(ii) [**5 marks**] Let the function f be defined by:

$$f(x) = \begin{cases} x^2 & -1 < x \leq 0 \\ x+1 & 0 < x \leq 1 \\ f(x+2) & \text{for all } x. \end{cases}$$

(a) Sketch the function for $-5 \leq x \leq 5$.
(b) Evaluate $f(-7.2)$ and $f(43.7)$.

2. [**10 marks**] An open box is made from a rectangular piece of tin 10m by 8m by cutting square pieces from each corner and folding up the sides. Each square has side length x.

(i) Draw a diagram of the box, labeling its height, breadth and width in terms of x.
(ii) What value of x produces the maximum volume of the box? Justify your answer.
(iii) What is this maximum volume?

3. [**8 marks**] A curve C is defined by the equation $y = (x-2)^{\frac{2}{3}} + 2$.

(i) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. (*Hint: what happens to these when $x = 2$?*)
(ii) Find all values of x for which y is increasing, and similarly for decreasing.
(iii) Show that the graph C is concave down for all x (except $x = 2$).
(iv) Using the above (and any other) information, sketch the curve C.

4. [**8 marks**] The following table gives the values of a function g:

x	0	5	10	15	20	25	30
$g(x)$	4	6.6	10.9	17.9	29.6	48.7	80.3

(i) Copy the table and make another row containing the values of $\ln[g(x)]$ for $x = 0,\ 5,\ \ldots,\ 30$.

(ii) Plot x against $\ln[g(x)]$ on graph paper.

(iii) Verify that x and $g(x)$ are related by an equation of the form $g(x) = Ae^{kx}$ by finding appropriate values for A and k.

5. (i) [**6 marks**] The function given by $f(x, y) = xy\cos(xy)$ has a critical point at $(0, 0)$. Is it a local minimum, local maximum or a saddle point?

(ii) [**8 marks**] A heated metal plate P is represented by a rectangle in the xy-plane, with vertices $(0, 0)$, $(4, 0)$, $(4, 2)$ and $(0, 2)$. At each point (x, y) of P the temperature, in degrees Celsius, is given by

$$T(x, y) = 10x^2 - 20xy + 20y + 20.$$

Find the point(s) at which the temperature of the plate is greatest and find the greatest temperature.

6. (i) [**5 marks**]

(a) Using integration by parts, show that

$$\frac{1}{2}\int_0^a x^2 e^{-x}dx = 1 - \left(\frac{1}{2}a^2 + a + 1\right)e^{-a}.$$

(b) Hence use your knowledge of limits to evaluate

$$\frac{1}{2}\int_0^\infty x^2 e^{-x}dx.$$

(ii) [**6 marks**] Evaluate the following definite integrals:

(a)

$$\int_1^{10} t(t^2 - 1)^{\frac{3}{2}}dt;$$

(b)

$$\int_0^{\frac{\pi}{2}} \sin^7 x \cos x dx.$$

7. [**11 marks**] Evaluate the following sums:

(i)

$$\sum_{k=1}^{50}\left(3k^3+2k^2+k+1\right);$$

(ii)

$$\sum_{j=0}^{99}\left(4^j+4j\right)$$

(iii)

$$3+7+11+\cdots+407.$$

(iv)

$$4+\frac{4}{3}+\frac{4}{3^2}+\cdots+\frac{4}{3^n},$$

for some positive integer n. Hence find the value of the sum as $n \to \infty$.

8. [**6 marks**] A scientist believes that the following data satisfy the equation $y = a + bx$, where a and b are constants:

x	0	1	2	3	4
y	0	4	10	16	19

In fact, there are no values of a and b which make the data fit the proposed formula exactly. However, the scientist is content to settle for a "line of best fit".

(i) From the data in the table, find $\sum x$, $\sum x^2$, $\sum y$ and $\sum xy$.

(ii) Hence find the equation for the "line of best fit".

(iii) Using this linear model, estimate the value of y when $x = 5.5$.

9. [**14 marks**]

(i) The rate of flow $R(t)$, measured in megalitres per hour, of water along a stormwater drain t hours after heavy rain is given by:

$$R(t) := \begin{cases} 1.25(6t-t^2) & 0 \le t \le 6 \\ 0 & t > 6. \end{cases}$$

Calculate the total quantity of water that flows along the drain after heavy rain.

(ii) Using an appropriate integral, find the average value of $\frac{s}{1+s^2}$ for $0 \le s \le 10$.

(iii) Find the area enclosed between the curves $y = x^2 - 4x$ and $y = 2x - x^2$.

(iv) After t days, the amount A of algae growing in a large (but neglected) swimming pool is increasing at a rate of $3t\sqrt{t}$ grams per day.

(a) If there were 5 grams of algae initially, find a formula for A in terms of t.

(b) How much algae is present after 15 days?