INTERACTIONS

$$\frac{dP}{dt} = 2P + c_1 Q$$

$$\frac{dQ}{dt} = 3Q + c_2 P$$

c_1 and c_2 represent the interaction between species P and Q. The sizes of c_1 and c_2 tell you how strong the interaction is. The signs of c_1 and c_2 tell you if the interaction is beneficial or detrimental.

Competition

Both c_1 and c_2 are negative, each species is worse off because of the presence of the other species.

Co-operation

Both c_1 and c_2 are positive, each species is better off because of the other species.

Host / Parasite or Predator / Prey

If $c_1 > 0$ and $c_2 < 0$ then P is better-off, but Q is worse-off.

Likewise, if $c_1 < 0$ and $c_2 > 0$, Q benefits from the interaction, but P suffers.

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SOLVING SIMULTANEOUS D.E.'S

$$P' = 2P - Q \qquad (1)$$

$$Q' = 3Q - 2P \qquad (2)$$

We will solve by eliminating one of the functions.

Eliminate Q as follows:

First rearrange eq$^{\underline{n}}$ (1) to make Q the subject

$$Q = 2P - P' \qquad (3)$$

Differentiate both sides of this eq$^{\underline{n}}$ w.r.t. time

$$Q' = 2P' - P'' \qquad (4)$$

Substitute Q and Q' from 3 and 4 into the equation we haven't used yet (2)

$$2P' - P'' = 3(2P - P') - 2P \quad (5)$$

Notice: Q no longer appears
This is a ~~second-order~~ second-order DE.

Rearrange 5 into a standard form:

$$P'' - 5P' + 4P = 0$$

Characteristic eqn is

$$k^2 - 5k + 4 = 0$$

Solutions are $k_1 = 1$ and $k_2 = 4$

Thus, $P(t) = A_1 e^{t} + A_2 e^{4t}$

Differentiate to get

$$P'(t) = A_1 e^{t} + 4A_2 e^{4t}$$

Substitute P and P' into 3

$$Q = 2(A_1 e^{t} + A_2 e^{4t}) - (A_1 e^{t} + 4A_2 e^{4t})$$

$$= A_1 e^{t} - 2A_2 e^{4t}$$

The solution is

$$P(t) = A_1 e^{t} + A_2 e^{4t}$$
$$Q(t) = A_1 e^{t} - 2A_2 e^{4t}$$

If we had eliminated ~~P~~ instead of Q we would have obtained something that looks different

$$Q(t) = A_1 e^{t} + A_2 e^{4t}$$
$$P(t) = A_1 e^{t} - \tfrac{1}{2} A_2 e^{4t}$$

<u>A way to check:</u>

The original non-interacting species had relative growth rates of 2 and 3

When they interact the growth rates that appear in the exponentials are

1 and 4

Notice $2+3 = 1+4$.

Not a coincidence.

To get a particular solution, we need TWO pieces of information

Natural information to give is $P(0)$ and $Q(0)$

Consider $P(0)=100$ and $Q(0)=70$.

Take the general solutions and sub in $t=0$

$$P(0) = A_1 + A_2 = 100$$
$$Q(0) = A_1 - 2A_2 = 70.$$

Solve these simultaneously to find

$$A_1 = 90 \quad \text{and} \quad A_2 = 10.$$

Particular solutions are

$$P(t) = 90e^t + 10e^{4t}$$
$$Q(t) = 90e^t - 20e^{4t}$$

Species will become extinct when

$$90e^t = 20e^{4t}$$

$$\Rightarrow \quad \frac{9}{2} = \frac{e^{4t}}{e^t} = e^{3t}$$

Solve this $t = \frac{1}{3}\ln(9/2)$.