Solutions to Practice Questions for Quiz 1

Basic Knowledge

1. $P(t) = A\,e^{5t}$ where A is arbitrary.
2. $P(t) = A\,e^{\frac{2}{3}t}$ where A is arbitrary.
3. $Q(t) = A - \frac{4}{3}t$ where A is arbitrary.
4. $X(t) = A\,e^{-\frac{4}{3}t}$ where A is arbitrary.
5. $X_n = A\,5^n$ where A is arbitrary.
6. $P_n = A - 5n$ where A is arbitrary.
7. $Q_n = A\,(-\frac{2}{3})^n$ where A is arbitrary.
8. $P(t) = 2\,e^{\frac{2}{3}t}$.
9. $Q(t) = 7 - \frac{4}{3}t$.
10. $X(t) = e^{-\frac{4}{3}t}$.
11. $P_n = 3(\frac{5}{7})^n$.
12. $Q_n = 5(-\frac{2}{3})^n$.

    You do not need to show any working, you only need to write the final answer(s) in the box provided on the quiz paper.

13. Equilibrium condition is $2P_{eq} + 5 = 0$. There is only one solution which is $-\frac{5}{2}$.
14. Equilibrium condition is $Y_{eq}^2 + 3Y_{eq} + 2 = 0$. This quadratic has two solutions -1 and -2.
15. You may need to rearrange this quadratic $4Q_{eq}^2 + 3Q_{eq} = 1$ before solving it. The solutions are $\frac{-3\pm\sqrt{3^2+16}}{8}$ which is -1 and $\frac{1}{4}$.
16. Equilibrium condition is $0 = kX_{eq} + b$. There is only one solution $X_{eq} = -\frac{b}{k}$.
17. Equilibrium condition is $P_{eq} = 5P_{eq} - 3$. Solution is $\frac{3}{4}$.
18. Equilibrium condition is $3P_{eq} + 2P_{eq} + 5 = 0$. Solution is -1.
19. Equilibrium condition is $7Q_{eq} + Q_{eq}^2 - 4Q_{eq} = 10$ which can be rearranged to give $Q_{eq}^2 + 3Q_{eq} - 10 = 0$. This quadratic has two solutions 2 and -5.
20. Equilibrium condition is $2Y_{eq} + 3Y_{eq}^2 + 2Y_{eq} + 1 = 0$ which can be rearranged to give $3Y_{eq}^2 + 4Y_{eq} + 1 = 0$. The solutions are $\frac{-4\pm\sqrt{4^2-12}}{6}$ which is -1 and $-\frac{1}{3}$.

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Numerical Calculations

1. METHOD 1: If you remember the formula $\frac{\ln(2)}{k}$ you can use it to immediately see that the half-life is approximately 34.657.

   METHOD 2: Otherwise, use the condition $C(t) = \frac{1}{2}C(0)$ which is the same as

   $$Ae^{-kt} = \frac{1}{2}A.$$

   This can be rearranged to give $2 = e^{kt}$ and taking natural logs of both sides gives $\ln(2) = kt$.

   Thus the half-life is given by $t = \frac{\ln(2)}{k} \approx 34.657$.

2. General solution is $S(t) = Ae^{at}$. The two pieces of information given lead to the equation $350 = 100e^{20a}$ which is the same as $3.5 = e^{20a}$. Taking natural logs of both sides $\ln(3.5) = 20a$.

   Thus $a = \frac{1}{20}\ln(3.5) \approx 0.0626$.

3. General solution is $L_n = A\,r^n$. The two pieces of information give $350 = A\,r$ and $100 = A\,r^{13}$. Use the first condition to give $A = \frac{350}{r}$ and substitute this into the second condition to give $100 = 350\,r^{12}$.

   Thus $r = (\frac{100}{350})^{\frac{1}{12}} \approx 0.9$.

4. The particular solution is $D_n = 50(1.02)^n$. Want to find out when $50(1.02)^n > 500$ which is the same as $(1.02)^n > 10$. Take natural logs of both sides $n\ln(1.02) > \ln(10)$ or $n > 116.276$. The first term which exceeds 10 times the initial value is D_{117}.

5. $X_0 = 1$, $X_1 = 2$, $X_2 = 2 \times 2^2 = 8$, $X_3 = 2 \times 8^2 = 128$.

6. $X_0 = 20$, $X_1 = \frac{1}{5}(20 - 20) = 0$, $X_2 = \frac{1}{5}(0 - 20) = -4$, $X_3 = \frac{1}{5}(-4 - 20) = -\frac{24}{5}$.

7. METHOD 1: $P_0 = 5$, $P_1 = 6$, $P_2 = 7.2$, $P_3 = 8.64$, $P_4 = 10.368$, $P_5 = 12.4416$, $P_6 = 14.92992$, $P_7 \approx 17.9$, $P_8 \approx 21.5$, $P_9 \approx 25.8$, $P_{10} \approx 30.969$, $P_{11} \approx 37.15$, $P_{12} \approx 44.58$, $P_{13} \approx 53.5$. Thus P_{13} is the first term which exceeds 50.

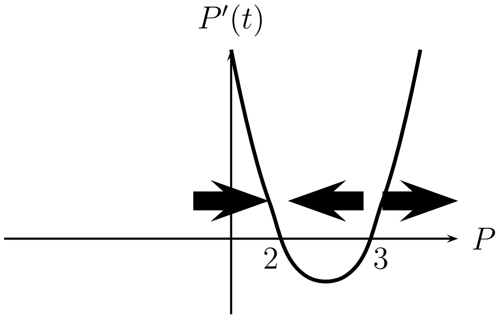
   METHOD 2: You can do this question using exactly the same method as in Q4 above. This gives the condition $(1.2)^n > 10$ or $n\ln(1.2) > \ln(10)$ or $n > 12.6$. Thus P_{13} is the first term which exceeds 50.

Stability

The solutions given in the stability section look very long because a variety of different ways of solving the problem are given.

You only need to be able to solve the problem in one way.

1. The equilibrium values are 2 and 3. A sketch of the RHS of this differential equation is a parabola passing through 2 and 3 on the horizontal axis.



METHOD 1:

The slope of the parabola is negative as it passes through 2, thus the equilibrium at 2 is STABLE. The slope of the parabola is positive as it passes through 3, thus the equilibrium at 3 is UNSTABLE.

METHOD 2:

The arrows show where the growth rate is positive (arrow to the right) or negative (arrow to the left). Inward arrows identify which equilibrium is stable and outward arrows which is unstable. Thus the equilibrium at 2 is stable and the equilibrium at 3 is unstable.

METHOD 3:

Alternatively, the DE has the form $P' = F(P)$ where $F(P) = P^2 - 5P + 6$. The derivative of this function is $F'(P) = 2P - 5$.

This is a *differential equation* so stability is determined by the *SIGN* of the derivative.

At $P_{eq} = 2$ we have $F'(2) = -1 < 0$, thus this equilibrium is STABLE. At $P_{eq} = 3$ we have $F'(3) = 1 > 0$, thus this equilibrium is UNSTABLE.

2. The equilibrium condition is $X_{eq} - X_{eq} = e^{-X_{eq}} - 0.3$ which can be rewritten $e^{-X_{eq}} = 0.3$. This has one solution which is $X_{eq} = -\ln(0.3) \approx 1.2$

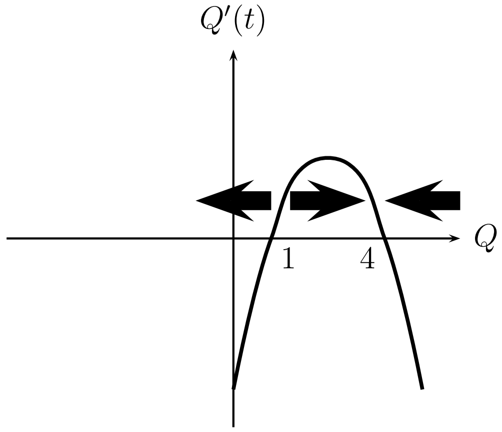
   Write the equation in the form $X_{n+1} = F(X_n)$ where $X_n = e^{-X_n} + X_n - 0.3$. The derivative of this function is $F'(X_n) = -e^{-X_n} + 1$ and at the equilibrium value the slope is $F'(-\ln(0.3)) = -e^{\ln(0.3)} + 1 = -0.3 + 1 = 0.7$.

   This is a *difference equation* so stability is determined by the *SIZE* of the derivative.

   Since $|0.7| < 1$ this equilibrium is STABLE.

3. The equilibrium condition is $5Q_{eq} = Q_{eq}^2 + 4$ which is a quadratic equation. Rearrange to give $Q_{eq}^2 - 5Q_{eq} + 4 = 0$ and the solutions are 1 and 4.

   The DE can be written in the form $Q' = F(Q)$ where $F(Q) = -Q^2 + 5Q - 4$. This is a parabola passing through 1 and 4 but is upside down when compared to the parabola in Q1.



   METHOD 1:

   The slope of the parabola is positive as it passes through 1, thus the equilibrium at 3 is UNSTABLE. The slope of the parabola is negative as it passes through 4, thus the equilibrium at 2 is STABLE.

   METHOD 2:

   The arrows show where the growth rate is positive (arrow to the right) or negative (arrow to the left). Inward arrows identify which equilibrium is stable and outward arrows which is unstable. Thus the equilibrium at 1 is unstable and the equilibrium at 4 is stable.

   METHOD 3:

   Alternatively, $F'(Q) = -2Q + 5$.

   This is a *differential equation* so stability is determined by the *SIGN* of the derivative.

   At $Q_{eq} = 1$ we have $F'(1) = 3 > 0$, thus this equilibrium is UNSTABLE. At $Q_{eq} = 4$ we have $F'(4) = -3 < 0$, thus this equilibrium is STABLE.

4. The equilibrium condition is $4Y_{eq} + Y_{eq} = 4 + Y_{eq}^2$ which can be rewritten $Y_{eq}^2 - 5Y_{eq} + 4 = 0$ which has solutions 1 and 4.

   This is a recurrence relation and it can be written in the form $Y_{n+1} = F(Y_n)$ where $F(Y_n) = \frac{1}{4}(Y_n^2 - Y_n + 4)$.

   Thus $F'(Y_n) = \frac{1}{4}(2Y_n - 1)$.

   This is a *difference equation* or *recurrence relation* so stability is determined by the *SIZE* of the derivative.

   $F'(1) = \frac{1}{4}$ and the magnitude is less than one, so this equilibrium is STABLE.

   $F'(4) = \frac{7}{4}$ and the magnitude is greater than one, so this equilibrium is UNSTABLE.

Logical thinking: Multiple Choice Questions

1. METHOD 1:

All of the options are about equilibrium, so first find all the equilibrium values.

The equilibrium condition is $X_{eq} = X_{eq}^3$ which can be rearranged to give $X_{eq}^3 - X_{eq} = 0$ which can be gradually factorised to give $X_{eq}(X_{eq}^2 - 1) = 0$ or $X_{eq}(X_{eq} - 1)(X_{eq} + 1) = 0$.

Thus there are three equilibrium values 0, -1 and +1.

The difference equation can be written in the form $X_{n+1} = F(X_n)$ where $F(X_n) = X_n^3$. Thus $F'(X_n) = 3X_n^2$.

Thus $F'(0) = 0$, $F'(1) = 3$ and $F'(-1) = 3$.

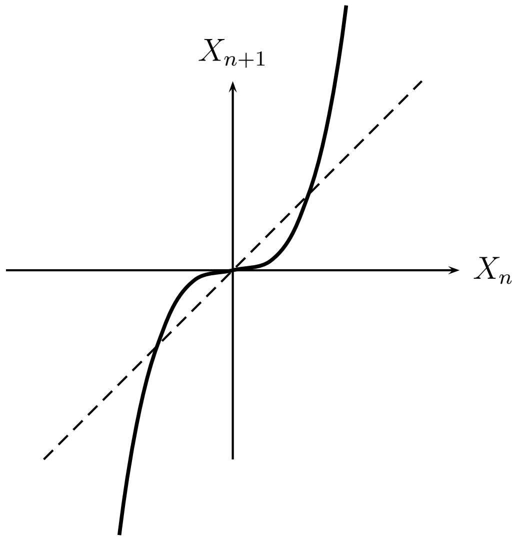
This is a *difference equation* or *recurrence relation* so stability is determined by the *SIZE* of the derivative.

$|F'(0)|$ is smaller than 1, the other two are both bigger than 1.

Thus option (c) is correct.

METHOD 2:

Alternatively you can sketch a cubic and a straight diagonal line.



There are clearly three equilibria in the sketch. The slope of the curve as it pass through the origin is clearly less than 1, so this equilibrium is stable, and the other two are unstable. Thus, again this leads to option (c) as the correct answer.

2. The equilibrium condition is $0 = 2P_{eq}^2 + bP_{eq} + 3$ and this is a quadratic equation. The question tells you that it has two different equilibriums solutions. Recall, that a quadratic has two solutions if the discriminant is positive.

The discriminant of this quadratic is $b^2 - 4 \times 2 \times 3 = b^2 - 24$. Thus $b^2 - 24 > 0$ if there are two solutions to the quadratic.

Thus option (c) is correct.

3. The equilibrium condition is $X_{eq} = rX_{eq}^2$ which can be rewritten as $rX_{eq}^2 - X_eq = 0$ which can be factorised $X_{eq}(rX_{eq} - 1) = 0$.

Thus there are two equilibrium values $X_{eq} = 0$ and $X_{eq} = \frac{1}{r}$. Thus, we can tell that option (a) is WRONG. *There are always two solutions.*

The recurrence relation can be written in the form $X_{n+1} = F(X_n)$ where $F(X_n) = rX_n^2$ and $F'(X_n) = 2rX_n$.

Thus $F'(0) = 0$ and $F'(\frac{1}{r}) = 2$.

This is a *difference equation* or *recurrence relation* so stability is determined by the *SIZE* of the derivative.

Thus one of the equilibria is stable and the other equilibrium is unstable. Thus (c) is TRUE and (d) is FALSE. The stability of the equilibrium at 0 does not have anything to do with the value of r so (b) is also FALSE.

The only correct option is (c).