Solutions to ASSIGNMENT 1 Due: 2pm, Monday, 30 January 2012

This assignment is worth 5% of the assessment for this unit of study.
Your assignment must be stapled with an assignment cover sheet inside a manilla folder with your name on it.

1. (a) A certain radioactive material decays at a rate proportional to the amount present. Initially there are 100 milligrams of the material present, and after two hours the material has lost 10% of its original mass.

(i) [3 marks] Find an expression for $X(t)$, the mass remaining in milligram at any time t (in hours), in terms of t.

(ii) [1 mark] Find the mass of the material after six hours.

(iii) [1 mark] Find the half-life of the material.

Solution:

(i) The differential equation is given by:

$$\frac{dX}{dt} = -kX, \text{ where } k > 0 \text{ is constant.}$$

The general solution is:

$$X(t) = Ae^{-kt}, \text{ where } A \text{ is an arbitrary constant.}$$

Now $X(0) = 100$ and $X(2) = 0.9 \times 100 = 90$. So $A = 100$ and $k = -\frac{1}{2}\ln 0.9$. Therefore

$$X(t) = 100e^{\frac{1}{2}t\ln 0.9}$$

(ii)

$$\begin{aligned} X(6) &= 100e^{\frac{1}{2}\cdot 6\ln 0.9} \\ &\approx 72.9 \end{aligned}$$

There are about 72.9 milligrams of the radioactive material remaining.

(iii) The half-life is $\frac{\ln 2}{|\frac{1}{2}\ln 0.9|} \approx 13.16$ hours.

(b) According to Newton's law of cooling, the rate of decrease of temperature of a body is proportional to the difference between its temperature and that of its environment. The temperature of the environment is 20°C and the body cools from 80°C to 60°C in 1 hour.

(i) [1 mark] Write the differential equation for $T(t)$, the temperature of the body (in °C) at time t (in hours).

(ii) [2 marks] Find the particular solution for the above differential equation.

(iii) [2 marks] Show that it will take somewhat over 4 hours to cool to 30°C.

Solution:

(i)

$$\frac{dT}{dt} = -k(T-20), \text{where } k > 0 \text{ is constant.}$$

(ii) The equilibrium, T_{eq} is 20. The general solution is

$$T(t) = Ae^{-kt} + 20, \text{ where } A \text{ is an arbitrary constant.}$$

We are given $T(0) = 80$ and $T(1) = 60$. So $T(0) = A + 20 = 80 \rightarrow A = 60$. $T(1) = 60e^{-k} + 20 = 60 \rightarrow k = \ln(\frac{3}{2})$.
Therefore

$$T(t) = 60e^{-\ln(\frac{3}{2})t} + 20$$

(iii)

$$\begin{aligned} T(t) &= 60e^{-\ln(\frac{3}{2})t} + 20 \\ 60e^{-\ln(\frac{3}{2})t} + 20 &= 30 \\ e^{-\ln(\frac{3}{2})t} &= \frac{1}{6} \\ \ln(\frac{3}{2})t &= \ln 6 \\ t &= \frac{\ln 6}{\ln(\frac{3}{2})} \\ &\approx 4.42 \end{aligned}$$

It takes about 4.42 hours to cool to 30°C.

2. A store has decided to start stocking a new brand of biscuits. They start with 200 biscuits and notice that after 1 month, half of the biscuit packets they stocked have been sold. They add 500 packets to their stock.

(a) [3 marks] Assuming such selling/restocking pattern continues, write down a difference equation for B_n, the number of biscuit packets n months after their introduction.

(b) [3 marks] Solve your difference equation to find an expression for B_n (in terms of n).

(c) [2 marks] According to your model, what do you expect to happen to biscuit packet number of the new brand? Justify your answer using your solution from (b).

(d) [2 marks] How many biscuit packets would they have after 4 months and after 12 months?

Solution:

(a) $B_{n+1} = \frac{1}{2}B_n + 500$

(b) The general solution is

$$B_n = A(\frac{1}{2})^n + 1000, \text{ where } A \text{ is an arbitrary constant.}$$

Now $B_0 = 200$ so

$$\begin{aligned} A(\frac{1}{2})^0 + 1000 &= 200 \\ A &= -800 \end{aligned}$$

Therefore

$$B_n = -800(\frac{1}{2})^n + 1000$$

(c) As $n \to \infty$, $(\frac{1}{2})^n \to 0$. Therefore as $n \to \infty$, the biscuit packet number of the new brand will approach 1000.

(d) $B_4 = -800(\frac{1}{2})^4 + 1000 = 950$ and $B_{12} = -800(\frac{1}{2})^{12} + 1000 = 999.8047$. Therefore there will be 950 biscuit packets after 4 months and 999 biscuit packets after 12 months.