Solutions to Practice Questions for Quiz 2

First Order Linear Equations

Give the general solution to these equations.

1. $P'(t) = 5P - 3$.
2. $3P' + 2P + 5 = 0$.
3. $P_{n+1} = 5P_n - 3$.
4. $3P_{n+1} + 2P_n + 5 = 0$.

Give the particular solution to these equations that satisfies the given condition.

5. $P'(t) = 7P - 4$ with $P(0) = 1$.
6. $2P' + 3P + 4 = 0$ with $P(0) = 0$.
7. $P_{n+1} = 3P_n - 5$ with $P_0 = 2$.
8. $2P_{n+1} + 5P_n - 14 = 0$ with $P_0 = 9$.

Solution:

1. $P(t) = Ae^{5t} + \frac{3}{5}$
2. $P(t) = Ae^{-\frac{2}{3}t} - \frac{5}{2}$
3. $P_n = A5^n + \frac{3}{4}$
4. $P_n = A(-\frac{2}{3})^n - 1$
5. $P(t) = \frac{3}{7}e^t + \frac{4}{7}$
6. $P(t) = \frac{4}{3}e^{-\frac{3}{2}t} - \frac{4}{3}$
7. $P_n = -\frac{1}{2}3^n + \frac{5}{2}$
8. $P_n = 7\left(-\frac{5}{2}\right)^n + 2$

Partial Fractions

Decompose these expressions into partial fractions and then integrate the result.

1. $\dfrac{2}{(x-3)(x-5)}$.
2. $\dfrac{11}{(x-4)(x+7)}$.
3. $\dfrac{1}{x^2+5x+6}$.

4. $\dfrac{5}{2x^2+7x+3}$

Solution:

1. $-\ln|x-3|+\ln|x-5|+C$
2. $\ln|x-4|-\ln|x+7|+C$
3. $\ln|x+2|-\ln|x+3|+C$
4. $-\ln|x+3|+\ln|2x+1|+C$

Separation of Variables

Find the functions $P(t)$ that solve each of the following problems. You must rearrange your solution to make P the subject.

1. $P'(t)=P^2t^3$ with $P(0)=1$.
2. $P'(t)=e^{-P}e^t$ with $P(0)=\ln(5)$.
3. $P'(t)=P^2e^t$ with $P(0)=1$.
4. $P'(t)=e^{-P}t^3$ with $P(0)=0$.

   Find the general solution to each of the following problems. In this case you do NOT need to rearrange your answer.

5. $P'(t)=(P^2-4)(t^2-4)$.
6. $P'(t)=\dfrac{P^2-4}{t^2-4}$.

Solution:

1. $P(t)=-\frac{4}{t^4-4}$
2. $P(t)=-\ln(e^t+4)$
3. $P(t)=-\frac{1}{e^t-2}$
4. $P(t)=\ln|\frac{1}{4}t^4+1|$
5. $\frac{1}{4}\ln\left|\frac{p-2}{p+2}\right|=\frac{t^3}{3}-4t+C$
6. $\frac{1}{4}\ln\left|\frac{p-2}{p+2}\right|=\frac{1}{4}\ln\left|\frac{t-2}{t+2}\right|+C$

Second Order Linear Equations

Find the general solution to the following second order linear equations

1. $P''(t) - 15P'(t) + 54P(t) = 0.$
2. $Q''(t) - 81Q(t) = 0.$
3. $P_{n+2} - 15P_{n+1} + 54P_n = 0.$
4. $Q_{n+2} - 81Q_n = 0.$
5. $x''(t) + 7x'(t) + 12x(t) = 4.$
6. $y''(t) - 2y'(t) - 15y(t) = 3.$
7. $x_{n+2} + 7x_{n+1} + 12x_n = 4.$
8. $y_{n+2} - 2y_{n+1} - 15y_n = 3.$

Solution:

1. $P(t) = A_1e^{6t} + A_2e^{9t}$
2. $Q(t) = A_1e^{-9t} + A_2e^{9t}$
3. $P_n = A_1 6^n + A_2 9^n$
4. $Q_n = A_1 9^t + A_2(-9)^t$
5. $x(t) = A_1e^{-3t} + A_2e^{-4t} + \frac{1}{3}$
6. $y(t) = A_1e^{-3t} + A_2e^{5t} - \frac{1}{5}$
7. $x_n = A_1(-3)^n + A_2(-4)^n + \frac{1}{5}$
8. $y_n = A_1 5^n + A_2(-3)^n - \frac{3}{16}$

Find solutions to the following equations that satisfy the given conditions

1. $x''(t) - 7x'(t) + 12x(t) = 0$ with $x(0) = 3$ and $x'(0) = 10$.
2. $x''(t) - 7x'(t) + 12x(t) = 12$ with $x(0) = 3$ and $x'(0) = 10$.
3. $y_{n+2} + 2y_{n+1} - 15y_n = 0$ with $y_0 = 3$ and $y_1 = 1$.
4. $y_{n+2} + 2y_{n+1} - 15y_n = 12$ with $y_0 = 3$ and $y_1 = 1$.

Solution:

1. $x(t) = 2e^{3t} + e^{4t}$
2. $x(t) = -2e^{3t} + 4e^{4t} + 1$
3. $y_n = (-5)^n + 2 \times 3^n$
4. $y_n = \frac{5}{4}(-5)^n + \frac{11}{4} \times 3^n - 1$

Iteration

1. Find the solution to $x = \frac{1}{2}\cos(x)$ to three decimal places.
2. Find the solution to $x = \frac{1}{10}e^x$ to three decimal places.
3. The equation $e^x + x = 9$ has a solution close to 2.

   Which of the following iteration schemes is useful in finding this solution?

   $$X_{n+1} = 9 - e^{X_n} \qquad \text{OR} \qquad X_{n+1} = \ln(9 - X_n)$$

   Use the correct scheme to find the solution to three decimal places.

Solution:

1. $x = 0.450$
2. $x = 0.112$
3. $X_{n+1} = \ln(9 - X_n), x = 1.953$

Logistic equation and logistic map

1. If $\ln(p) - \ln(1-p) = rt + C$ and $p(0) = \frac{1}{3}$ give an exact expression for C.
2. In Robert May's famous 1976 paper he looks at the equation $X_{n+1} = F(X_n)$ with

   $$F(X) = Xe^{r(1-X)}.$$

   For what range of values of r does this iteration have a stable positive fixed point?

Solution:

1. $C = -\ln 2$
2. $0 < r < 2$