Tutorial 10

Preparatory Questions

1. Write down the characteristic (or auxilliary) equations and hence find the general solution to the following second order linear differential equations

(a) $x''(t) + 7x'(t) + 12x(t) = 0.$

(b) $y''(t) - 2y'(t) - 15y(t) = 0.$

(c) $P''(t) - 15P'(t) + 54P(t) = 0.$

(d) $Q''(t) - 81Q(t) = 0.$

2. Write down the characteristic (or auxilliary) equations and hence find the general solution to the following second order linear difference equations

(a) $x_{n+2} + 7x_{n+1} + 12x_n = 0.$

(b) $y_{n+2} - 2y_{n+1} - 15y_n = 0.$

(c) $P_{n+2} - 15P_{n+1} + 54P_n = 0.$

(d) $Q_{n+2} - 81Q_n = 0.$

3. Find the general solution to the following equations (you may re-use the answers from Q1 and Q2 to save time and effort).

(a) $x''(t) + 7x'(t) + 12x(t) = 4.$

(b) $x_{n+2} + 7x_{n+1} + 12x_n = 4.$

(c) $y''(t) - 2y'(t) - 15y(t) = 3.$

(d) $y_{n+2} - 2y_{n+1} - 15y_n = 3.$

Tutorial Questions

4. Consider the problem $x''(t) - 7x'(t) + 12x(t) = 0$ with initial conditions $x(0) = 3$ and $x'(0) = 10$.

(a) Write down and solve the characteristic equation.

(b) Thus, write down the general solution.

(c) Differentiate the general solution.

(d) Write down the two equations that come from the initial conditions.

(e) Find the arbitrary constants by solving the resulting simultaneous equations.

(f) Thus, show that the particular solution is $x(t) = 2e^{3t} + e^{4t}$.

5. Repeat the previous question for the problem $x''(t) - 7x'(t) + 12x(t) = 12$ with initial conditions $x(0) = 3$ and $x'(0) = 10$. Think carefully about which steps give the same answers and which steps give different answers.

6. Consider the problem $y_{n+2} + 2y_{n+1} - 15y_n = 0$. with initial conditions $y_0 = 3$ and $y_1 = 1$.

   (a) Write down and solve the characteristic equation.

   (b) Thus, write down the general solution.

   (c) Write down the two equations that come from the initial conditions.

   (d) Find the arbitrary constants by solving the resulting simultaneous equations.

   (e) Thus, show that the particular solution is $y_n = (-5)^n + 2(3)^n$.

7. Repeat the previous question for the problem $y_{n+2} + 2y_{n+1} - 15y_n = 12$ with initial conditions $y_0 = 3$ and $y_1 = 1$. Think carefully about which steps give the same answers and which give different answers.

Advanced

8. Solve the problem $P''(t) - 3P'(t) = 0$ with initial conditions $P(0) = 5$ and $P'(0) = 3$.

   [Hint: The characteristic equation is $k^2 - 3k = 0$ and the solution is $P(t) = 4 + e^{3t}$. There are still two exponential functions in this solution, it's just that one is a bit hard to notice!]

9. What is wrong with the following question?

   Solve the problem $P_{n+2} - 3P_{n+1} = 0$ with initial conditions $P_0 = 5$ and $P_1 = 3$.

Just for Fun

10. You want to design an exam question about second order differential equations so that the answer comes out to be $P(t) = 2e^{3t} + 3e^{4t}$.

   (a) Calculate $P'(t)$.

   (b) Working backwards, decide what the initial conditions $P(0)$ and $P'(0)$ should be.

   (c) Find the simplest quadratic equation that has roots 3 and 4.

   (d) Then, use this quadratic to write down a second-order DE that has general solution $P(t) = A_1 e^{3t} + A_2 e^{4t}$.

   (e) Write out a version of what the question should say.

Partial solutions and/or hints to some of the preparatory questions:

1(b) $y(t) = A_1 e^{-3t} + A_2 e^{5t}$.
1(d) $Q(t) = A_1 e^{-9t} + A_2 e^{9t}$.
2(a) $x_n = A_1(-3)^n + A_2(-4)^n$.
2(c) $P_n = A_1 6^n + A_2 9^n$.
3(a) $x(t) = \frac{1}{3} + A_1 e^{-3t} + A_2 e^{-4t}$.
3(b) $x_n = \frac{1}{5} + A_1(-3)^n + A_2(-4)^n$.
3(c) $y(t) = -\frac{1}{5} + A_1 e^{-3t} + A_2 e^{5t}$.
3(d) $y_n = -\frac{3}{16} + A_1(-3)^n + A_2 5^n$.