Solutions to Tutorial 1

Preparatory Questions

1. First write down the general solution to each of the following differential equations. (You may need to rearrange the equation first to get it into a form you recognise.) Then calculate the particular solution that satisfies the given condition.

(a) $3\dfrac{dP}{dt} = P$ with $P(0) = 2$.

(b) $3Q'(t) + 4 = 0$ with $Q(0) + Q(3) = 10$.

(c) $3X' + 4X = 0$ with $X'(0) = 1$.

(d) $aY' - bY = 0$ with $Y(1) = 4$.

(e) $\dfrac{dA}{dt} = c - d$ with $A(1) = c$.

Solution: *Remember that you can use any letter you like (uppercase or lowercase) for the arbitrary constant provided it has not been used for something else.*

(a) Rearrange first into the form $\dfrac{1}{P}\dfrac{dP}{dt} = \dfrac{1}{3}$ or if you prefer, into the form $P'(t) = \frac{1}{3}P$. Thus, you can now see that the relative growth rate is $\frac{1}{3}$. The general solution is $P(t) = A\,e^{\frac{1}{3}t}$ where A is arbitrary. Thus $P(0) = A = 2$ and the particular solution is $P(t) = 2\,e^{\frac{1}{3}t}$.

(b) Rearrange first $Q'(t) = -\dfrac{4}{3}$. General solution is $Q(t) = -\frac{4}{3}t + C$ where C is arbitrary. Thus $Q(0) + Q(3) = 2C - 4 = 10$. Hence, $C = 7$ and the particular solution is $Q(t) = 7 - \frac{4}{3}t$.

(c) Rearrange first $X'(t) = -\dfrac{4}{3}X$. General solution is $X(t) = A\,e^{-\frac{4}{3}t}$ where A is arbitrary. Differentiating gives $X'(t) = -\frac{4}{3}A\,e^{-\frac{4}{3}t}$ and thus $X'(0) = -\frac{4}{3}A$. Thus, $A = -\frac{3}{4}$ and the particular solution is $X(t) = -\frac{3}{4}e^{-\frac{4}{3}t}$.

(d) Rearrange first $Y'(t) = \dfrac{b}{a}Y$. General solution is $Y(t) = A\,e^{\frac{b}{a}t}$ where A is arbitrary. Thus $Y(1) = A\,e^{\frac{b}{a}} = 4$. Thus, $A = 4e^{-\frac{b}{a}}$ and the particular solution is $Y(t) = 4e^{-\frac{b}{a}}e^{\frac{b}{a}t}$

(e) General solution is $A(t) = (c - d)t + b$ where b is arbitrary. Now $A(1) = c - d + b$ thus $c = c - d + b$ which means $b = d$. The particular solution is $A(t) = (c - d)t + d$. *Remember that the condition is used to determine the value of the arbitrary constant in terms of information already given.*

2. Use your favourite search engine to investigate the models associated with the names below. For each model try to find out what field it is used in, how many differential equations are used in the model and what the order is of each equation.

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(a) Hindmarsh-Rose

(b) Kermack-McKendrick

(c) Hodgkin-Huxley

(d) Lotka-Volterra

(e) Verhulst

(f) Weber-Fechner

(g) Morris-Lecar

(h) Della Giusta, Di Tommaso & Strom

Solution: This question was to give you a feel for what areas of biology and medicine use these types of differential equations. We will study some of these later.

(a) The Hindmarsh-Rose model is used to model neuronal activity using 3 equations each of which is first-order.

(b) The Kermack-McKendrick model is used to model epidemics. It is also called the SIR model, and uses 3 equations each of which is first-order.

(c) The Hodgkin-Huxley model is used to model the action potentials in neurons. There is a single first-order differential equation for the voltage and several other differential equations for related quantities.

(d) The Lotka-Volterra equations are used to model the interaction of predators and their prey. There are two equations each of which is first-order.

(e) The Verhulst equation was first used to describe resource-limited population growth, but has since been applied to many, many fields and is also called the logistic equation. It is a single differential equation of first-order.

(f) The Weber-Fechner law is used in psychology to relate stimuli and perceptions. It is a single first-order differential equation.

(g) The Morris-Lecar model describes neurons (often nerves in muscles) and is a pair of equations each of which is first-order.

(h) Della Giusta, Di Tommaso & Strom are economists who have used differential equations to describe the sex-trade industry (mostly in third-world countries). One of their papers can be found in the journal of population economics

http://www.springerlink.com/content/k2v47x73u2n62604/fulltext.pdf

and uses maths beyond the scope of this course. A criticism of their work can be found at http://www.guardian.co.uk/science/2007/mar/13/highereducation.research

3. Practise your differentiation skills by completing the table below. The table is used in Q8.

$f(t)$	$f'(t)$	$f''(t)$	$f'''(t)$
$t^3 - 3t + 2$			
$\cos(t)$			
$\ln(t) + t^2$			$\frac{2}{t^3}$
$\sin(t) + e^t$			
te^t	$e^t + te^t$		
$e^t \sin(t)$			$2e^t \cos(t) - 2e^t \sin(t)$

Solution:

$f(t)$	$f'(t)$	$f''(t)$	$f'''(t)$
$t^3 - 3t + 2$	$3t^2 - 3$	$6t$	6
$\cos(t)$	$-\sin(t)$	$-\cos(t)$	$\sin(t)$
$\ln(t) + t^2$	$\frac{1}{t} + 2t$	$-\frac{1}{t^2} + 2$	$\frac{2}{t^3}$
$\sin(t) + e^t$	$\cos(t) + e^t$	$-\sin(t) + e^t$	$-\cos(t) + e^t$
te^t	$e^t + te^t$	$2e^t + te^t$	$3e^t + te^t$
$e^t \sin(t)$	$e^t \sin(t) + e^t \cos(t)$	$2e^t \cos(t)$	$2e^t \cos(t) - 2e^t \sin(t)$

Tutorial Questions

4. Radioactive carbon-14 is present in small quantities in all living matter, and is constantly replenished from the atmosphere. When an organism dies, the carbon-14 decays to stable carbon-12. Suppose that $C(t)$ is the mass of carbon-14 (measured in grams) present in a sample of bone at time t (measured in years). The radioactive decay is described by the differential equation

$$\frac{dC}{dt} = -kC.$$

(a) If the half-life of the decay is 5730 years calculate the relative rate of decay k to three significant figures.

(b) In January this year, a team of scientists found a human bone fragment which contained 89% of the amount of carbon-14 contained in living human bone. How old is the bone fragment?

Solution:

(a) The general solution to this differential equation is

$$C(t) = A\, e^{-kt}.$$

The half-life of carbon-14 is 5730 years which means that

$$C(5730) = \frac{1}{2} C(0).$$

Using the general solution this means

$$A\,e^{-5730k} = \frac{1}{2}A.$$

Solving for k (by taking logs) gives

$$-5730k = \ln(\frac{1}{2})$$

or

$$k = \frac{\ln(2)}{5730} \approx 0.000121 = 1.21 \times 10^{-4}$$

to three significant figures.

(b) In January this year, $C(t) = 0.89\,C(0)$, where t is the age of the bone fragment. Thus, substituting in from the general solution gives

$$A\,e^{-0.000121t} = 0.89\,A.$$

Solving for t gives

$$-0.000121t = \ln(0.89)$$

or

$$t \approx 963$$

So the bone is, to three significant figures, 963 years old.

5. Uranium-234 occurs in most natural sources of water, and any material that precipitates or is grown from such waters also contains trace amounts. When uranium-234 decays it turns into thorium-230 which is not soluble. Thus measuring the composition of uranium-thorium is a method suitable for dating carbonate deposits such as coral reefs, shells and limestone formations. The radioactive decay can be described by the differential equation $Q' = -k\,Q$ where $Q(t)$ is the amount of uranium-234 in grams and $k = 2.83 \times 10^{-6}$(measured in years).

(a) Find the particular solution for $Q(t)$ that corresponds to a sample initially containing 100 g of uranium-234.

(b) What is the half-life of this process?

(c) How long does it take for a sample of uranium-234 to be reduced to $\frac{1}{8}$ of its original amount?

Solution:

(a) The general solution is $Q(t) = A\,e^{-2.83\times10^{-6}t}$ where A is arbitrary. If $Q(0) = 100$ then this means $A = 100$ so the particular solution is $Q(t) = 100e^{-2.83\times10^{-6}t}$.

(b) Using the formula from the notes, the half-life is

$$t = \frac{\ln 2}{2.83 \times 10^{-6}} = 2.44928 \times 10^{5}$$

or about 245,000 years. If you don't remember the formula you can solve the equation

$$\frac{1}{2}A = A\,e^{-2.83\times10^{-6}t}$$

to find t.

(c) Each half-life reduces the amount by a factor of 2, thus after 3 half-lifes the amount will be reduced by $8 = 2^3$. The time required is, thus, $3 \times 245,000 = 735,000$ years.

6. During the 1988 Olympics in Seoul, Australian fencer Alex Watson was banned for elevated levels of caffeine. The allowable limit was 12 μg/ml and Alex was found to have a concentration of 14 μg/ml. Assuming the half-life of caffeine is 4 hours, how long would it have taken for Alex's caffeine level to drop below the allowable limit.

Solution: Let $C(t)$ be the amount of caffeine. Then

$$C(t) = C(0)e^{-kt}$$

where t is measured in hours and

$$k = \frac{\ln(2)}{4} \approx 0.173.$$

The time taken for the caffeine level to drop from 14 to 12 is given by the equation

$$12 = 14\, e^{-kt}.$$

Taking logs of both sides gives

$$\ln(12) = \ln(14) - kt$$

or

$$t = \frac{\ln(14) - \ln(12)}{k} \approx 0.89$$

and 0.89 hours is about 53 minutes.

7. A Long-Island iced-tea is a deadly cocktail that usually contains 60 grams of alcohol (or the equivalent of 6 standard drinks). Let $C(t)$ be the amount of alcohol (measured in grams) at time t after the cocktail is consumed (where t is measured in hours). If alcohol is metabolised at a constant rate of 10 grams per hour, write down the differential equation satisfied by $C(t)$. Give the particular solution corresponding to an initial amount of 60 grams. If someone drinks only a Long-Island iced-tea at 7pm, and then stops drinking for the rest of the night, at what time is it legally safe for them to drive a car. (Assume two standard drinks is within the legal limit for driving.)

Solution: The differential equation is $C'(t) = -a$ where $a = 10$ if C is measured in grams and t is measured in hours. The solution is $C(t) = 60 - 10t$. Two standard drinks is 20 grams of alcohol, and thus the equation that needs to be solved is $C(t) = 20$. The answer is $t = 4$. As expected, the person has to wait 4 hours to reduce their alcohol content back down to the equivalent of two standard drinks. They need to wait until at least 11 pm before driving.

8. Use the derivatives in the table from Q3 to decide which functions satisfy which differential equation(s) from the list below. *You are not expected to solve these equations, merely check if certain functions are solutions or not by substituting the derivatives into the differential equation.*

(i) $f'''(t) - f''(t) + f'(t) - f(t) = 0$

(ii) $f'(t)^2 + f(t)^2 = 1$

(iii) $f''(t) - 2f'(t) + f(t) = 0$

(iv) $t^2 f''(t) - t f'(t) + 2 = 0$

(v) $(t+2)(t-1)f'(t) = 3(t+1)f(t)$

(vi) $f''(t) - 2f'(t) + 2f(t) = 0$

A function may satisfy more than one equation; and an equation may be satisfied by more than one function.

Solution:

- $t^3 - 3t + 2$ is a solution to eqn (v)
- $\cos(t)$ is a solution to eqn (i) and (ii)
- $\ln(t) + t^2$ is a solution to eqn (iv)
- $\sin(t) + e^t$ is a solution to eqn (i)
- te^t is a solution to eqn (iii)
- $e^t \sin(t)$ is a solution to eqn (vi)