Tutorial 2

Preparatory Questions

1. Find the equilibrium (steady-state) solutions to each of the following differential equations.

(a) $3\dfrac{dP}{dt} + 2P + 5 = 0.$

(b) $7Q' + 4Q^2 + 3Q = 1.$

(c) $5\dfrac{dX}{dt} + 4\tan(X) = 0.$

(d) $aY'' + b(Y')^2 + cY + d = 0.$

2. Each of the following differential equations has the form $X'(t) = F(X)$. Sketch the function $F(X)$ vs. X for all X and thus find the equilibrium solutions and use your sketch to decide which equilibria are stable and which are unstable. Some of the differential equations may have no equilibria at all.

(a) $X'(t) = 3 - 2X.$

(b) $X'(t) = X^2 - 5X + 6.$

(c) $X'(t) = e^X.$

(d) $X'(t) = e^X - 3.$

Tutorial Questions

3. According to the United Nations, declining birth rates and reduced fertility are a problem for many industrialised countries. Japan and many nations in Western Europe today would have declining populations if it were not for international immigration. Consider the following model for the population $P(t)$ as a function of time

$$\frac{dP}{dt} = -kP + a$$

where k and a are positive constants. The first term $-kP$ represents negative relative growth (decay) and the second term a represent a constant immigration rate.

Does this model have a positive equilibrium population, and is this equilibrium stable?

4. Some developing countries have a different problem to that in the previous question. For these countries the relative growth rate is positive, and there is a significant amount of emigration. Consider the following model for the population $P(t)$ as a function of time

$$\frac{dP}{dt} = kP - a$$

where k and a are positive constants. The first term kP represents positive relative growth and the second term $-a$ represent a constant emigration rate.

Does this model have a positive equilibrium population, and is this equilibrium stable?

5. (a) Sketch the graphs of the $y = x^3$ and $y = 5 - 2x$ on one diagram. [Hint: If you have trouble with the sketch calculate a few points that you know will lie on each graph. For example $(1, 3)$ lies on the straight line.]

(b) What do the intersection(s) of the graphs tell you about the number of solutions to the equation $x^3 + 2x - 5 = 0$?

(c) (Advanced part) Where are the intersection point(s) approximately located?

(d) Use this information to answer the following question: How many equilibrium solutions does the differential equation

$$\frac{dP}{dt} = P^3 + 2P - 5$$

have, and which ones are stable? You are *not* required to say *exactly* where the equilibrium solution(s) are.

6. How many equilibrium solutions does the DE

$$\frac{dQ}{dt} = \sin(Q) - Q + 1$$

have and what is their stability? [Hint: Sketch the graphs of $y = \sin(x)$ and $y = x - 1$ on the same diagram.]

7. Use a sketch to find all equilibrium solutions of the DEs below and decide which equilibria are stable and which are unstable.

(a) $X'(t) = \tan(X)$.

(b) $X'(t) = \tan(X) - 1$.

8. Some extremely primitive aquatic organisms absorb nutrients through their skin, and the rate at which this occurs is proportional to their surface area. On the other hand, their need for nutrients is related to their volume. Thus their growth is affected by two quantities: a positive contribution from their surface area and a negative contribution from their volume. If L represents the length of such an organism then a simple model of growth might be

$$\frac{dL}{dt} = aL^2 - bL^3$$

where the first term is proportional to surface area, and the second term is proportional to volume. The two constants a and b are both positive and depend on details of the biology.

How many equilibria does the model have? Does this model have a positive equilibrium?

(Advanced Question) Is the positive equilibrium stable? [Hint: You may want to first show that the sign of $\frac{dL}{dt}$ is the same as the sign of $a - bL$.]