Solutions to Tutorial 2

Preparatory Questions

1. Find the equilibrium (steady-state) solutions to each of the following differential equations.

(a) $3\dfrac{dP}{dt} + 2P + 5 = 0.$

(b) $7Q' + 4Q^2 + 3Q = 1.$

(c) $5\dfrac{dX}{dt} + 4\tan(X) = 0.$

(d) $aY'' + b(Y')^2 + cY + d = 0.$

Solution: In each case, set all derivatives to zero to find the equilibrium condition, and then find the solutions of that equation.

(a) The equilibrium condition is $2P_{eq} + 5 = 0$. There is only one solution $P_{eq} = -\frac{5}{2}$.

(b) The equilibrium condition is $4Q_{eq}^2 + 3Q_{eq} = 1$. This is a quadratic equation and it has two solutions which are $Q_{eq} = -1$ and $Q_{eq} = \frac{1}{4}$.

(c) The equilibrium condition is $4\tan(X_{eq}) = 0$. There are an infinite number of solutions, which are all the multiples of π.

(d) The equilibrium condition is $cY_{eq} + d = 0$. There is only one solution $Y_{eq} = -\frac{d}{c}$.

2. Each of the following differential equations has the form $X'(t) = F(X)$. Sketch the function $F(X)$ vs. X for all X and thus find the equilibrium solutions and use your sketch to decide which equilibria are stable and which are unstable. Some of the differential equations may have no equilibria at all.

(a) $X'(t) = 3 - 2X.$

(b) $X'(t) = X^2 - 5X + 6.$

(c) $X'(t) = e^X.$

(d) $X'(t) = e^X - 3.$

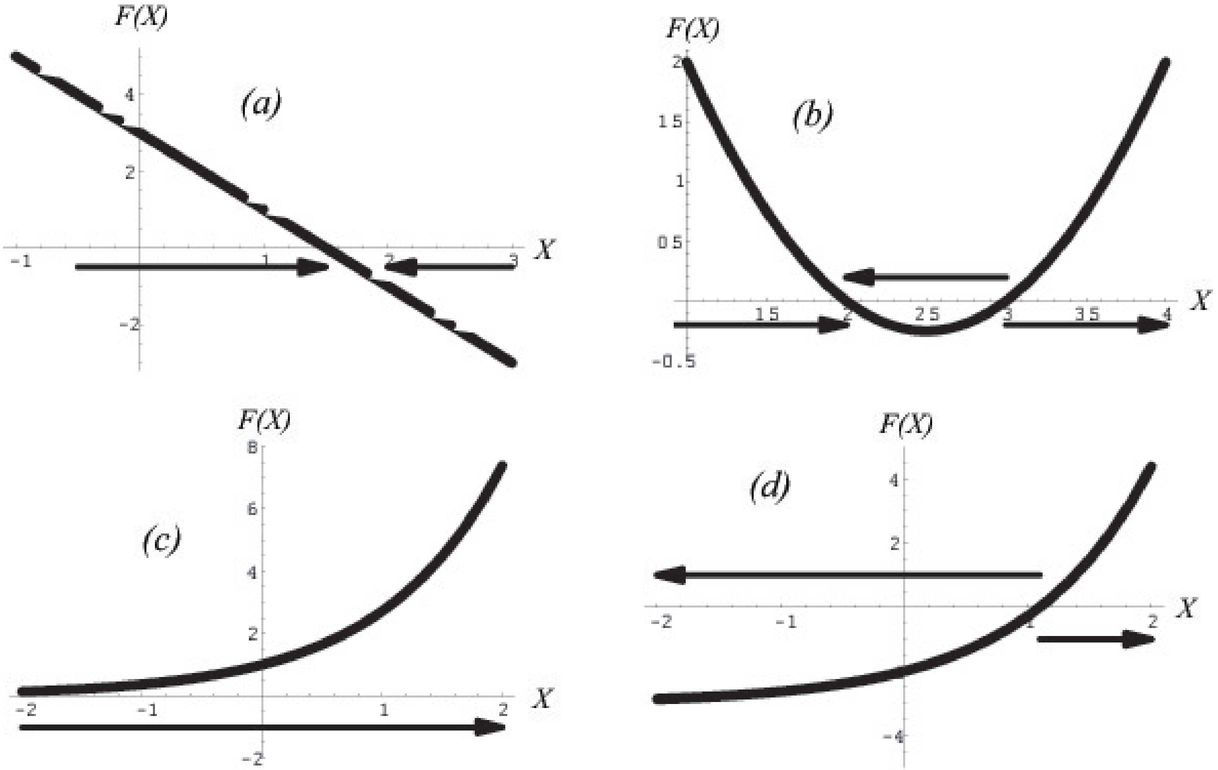
Solution: See sketches on next page. Your sketches don't need to be this accurate.

(a) This is a straight line, slope -2, vertical-intercept 3 and horizontal-intercept 3/2. There is one equilibrium solution at $X_{eq} = \frac{3}{2}$. The arrow to the left of the equilibrium is pointing right, the other arrow is pointing left. Thus the equilibrium is stable.

(b) This is a parabola passing through the values 2 and 3 on the x-axis. Thus, there are two equilibrium solutions at $X_{eq} = 2$ and $X_{eq} = 3$. The arrow in the middle region (between 2 and 3) points left, the other arrows point right. Thus $X_{eq} = 2$ is stable, and $X_{eq} = 3$ is unstable.

(c) This graph crosses the y-axis at 1 and is always above the x-axis. Thus the growth rate is always positive. Thus there are no equilibria.

(d) This is the same graph as in (c), BUT it has been shifted down by 3. This means it will now cross the vertical axis at -2 (rather than 1) and it will cross the horizontal axis at the point where $e^X = 3$ which is $\ln(3)$. Thus there is one equilibrium at $X_{eq} = \ln(3) \approx 1.099$. The arrow to the left points left, and the arrow on the right points right, so the equilibrium is unstable.



Tutorial Questions

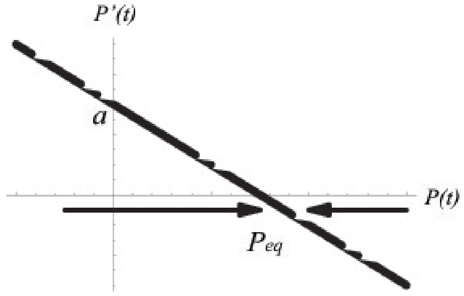
3. According to the United Nations, declining birth rates and reduced fertility are a problem for many industrialised countries. Japan and many nations in Western Europe today would have declining populations if it were not for international immigration. Consider the following model for the population $P(t)$ as a function of time

$$\frac{dP}{dt} = -kP + a$$

where k and a are positive constants. The first term $-kP$ represents negative relative growth (decay) and the second term a represent a constant immigration rate.

Does this model have a positive equilibrium population, and is this equilibrium stable?

Solution: The equilibrium condition is $0 = -kP_{eq} + a$ which has a single solution $P_{eq} = a/k$. Since a and k are both positive, then the equilibrium population is also positive.



A sketch of the function $F(P) = -kP + a$ is a straight line with negative slope, passing through a on the vertical axis and P_{eq} on the horizontal axis. If $P < P_{eq}$ then the effect of immigration (the term a) outweighs the negative growth ($-kP$) and the population increases. On the other hand, if $P > P_{eq}$ then the effect of immigration (the term a)

is smaller than the negative growth ($-kP$) and the population decreases. Together this means that the equilibrium is stable.

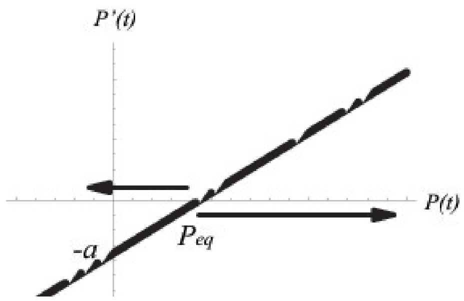
4. Some developing countries have a different problem to that in the previous question. For these countries the relative growth rate is positive, and there is a significant amount of emigration. Consider the following model for the population $P(t)$ as a function of time

$$\frac{dP}{dt} = kP - a$$

where k and a are positive constants. The first term kP represents positive relative growth and the second term $-a$ represent a constant emigration rate.

Does this model have a positive equilibrium population, and is this equilibrium stable?

Solution: The equilibrium condition is $0 = kP_{eq} - a$ which has a single solution $P_{eq} = a/k$. Since a and k are both positive, then the equilibrium population is also positive.



A sketch of the function $F(P) = kP - a$ is a straight line with positive slope, passing through $-a$ on the vertical axis and P_{eq} on the horizontal axis. If $P < P_{eq}$ then the effect of emigration (the term a) outweighs the growth (kP) and the population decreases. On the other hand, if $P > P_{eq}$ then the effect of emigration (the term a) is smaller than the growth (kP) and the population increases. Together, these two facts imply that the equilibrium is unstable.

5. (a) Sketch the graphs of the $y = x^3$ and $y = 5 - 2x$ on one diagram. [Hint: If you have trouble with the sketch calculate a few points that you know will lie on each graph. For example $(1, 3)$ lies on the straight line.]

(b) What do the intersection(s) of the graphs tell you about the number of solutions to the equation $x^3 + 2x - 5 = 0$?

(c) (Advanced part) Where are the intersection point(s) approximately located?

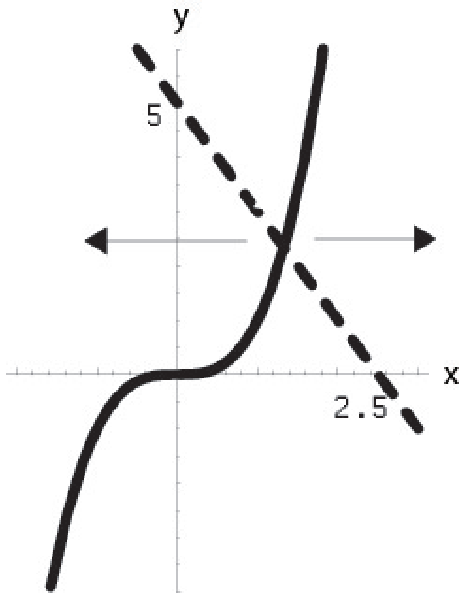
(d) Use this information to answer the following question: How many equilibrium solutions does the differential equation

$$\frac{dP}{dt} = P^3 + 2P - 5$$

have, and which ones are stable? You are *not* required to say *exactly* where the equilibrium solution(s) are.

Solution:

(a) The sketch shows $y = x^3$ as a solid curve, and $y = 5 - 2x$ as a dashed line. The line has intercept 5 on the vertical axis, and crosses the horizontal axis at $\frac{5}{2}$. To help with the sketch you can calculate a few points that you know must lie on the curves. For example, $(1, 1)$ and $(2, 8)$ must lie on the cubic curve; and $(1, 3)$ and $(2, 1)$ must lie on the straight line.



(b) The sketch reveals that the two graphs only cross once. Thus, there is only one solution to the equation.

(c) This intersection point has an x-value somewhere between 0 and $\frac{5}{2}$. You don't need a particularly accurate sketch to deduce this much information. If you plot the points mentioned in part (a) on the curves you can even tell that this intersection point has an x-value somewhere between 1 and 2.

(d) The equilibrium values of the differential equation are given by the solutions of

$$0 = P_{eq}^3 + 2P_{eq} - 5.$$

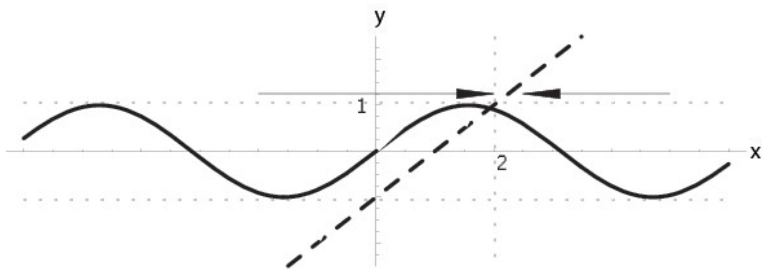
Form the previous parts of this question, we know that there is only one equilibrium (and it is somewhere between 0 and $\frac{5}{2}$). Looking at the sketch again, we can see that to the right of the equilibrium the cubic graph is *above* the straight line. This means that $\frac{dP}{dt}$ is positive and an arrow is drawn in this region pointing to the right. Likewise, to the left of equilibrium, growth is negative so an arrow is drawn pointing to the left. This reveals that the equilibrium is unstable.

6. How many equilibrium solutions does the DE

$$\frac{dQ}{dt} = \sin(Q) - Q + 1$$

have and what is their stability? [Hint: Sketch the graphs of $y = \sin(x)$ and $y = x - 1$ on the same diagram.]

Solution: The sketch shows $y = \sin(x)$ as a solid curve, and $y = x - 1$ as a dashed line. The line has intercept -1 on the vertical axis, and crosses the horizontal axis at 1. Two horizontal dotted lines are shown at $y = \pm 1$, to help show that the sine function never goes outside the range $[-1, 1]$. A vertical dotted line is also shown at $x = 2$. To



help with the sketch you can calculate a few points that you know will be on each graph. For example, the point $(2, 1)$ lies on the straight line and the point $(\frac{\pi}{2}, 1)$ lies on the trig. function. The sketch reveals that there is only one equilibrium solution. The point where the two graphs cross is to the right of $\frac{\pi}{2}$ but to the left of 2. To the right of the crossing point, the straight line is higher than the trig function. This means that $Q - 1 > \sin(Q)$ if $Q > Q_{eq}$, so in this region the growth rate will be negative, and the arrow is drawn pointing left. Likewise, on the other side of the equilibrium, $Q - 1 < \sin(Q)$ if $Q < Q_{eq}$ and the growth rate is positive, thus the arrow points right. Together, the two arrows reveal that the equilibrium is stable.

7. Use a sketch to find all equilibrium solutions of the DEs below and decide which equilibria are stable and which are unstable.

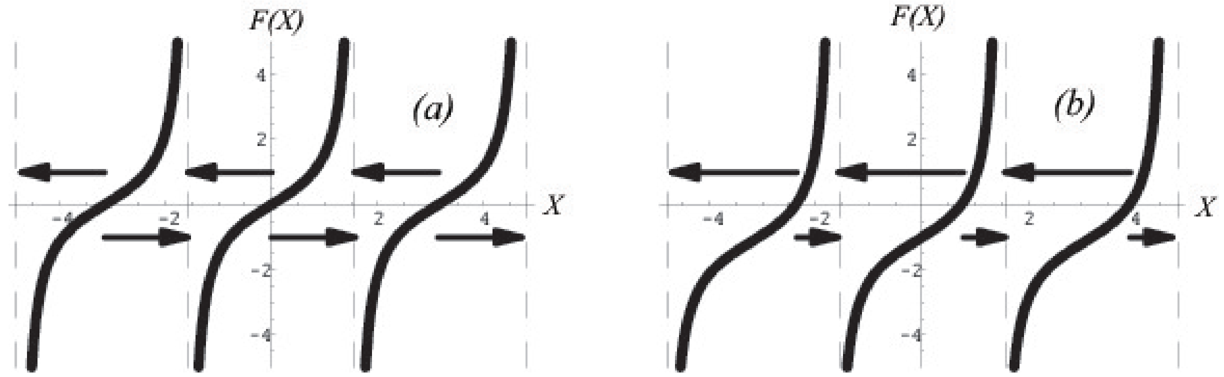
(a) $X'(t) = \tan(X)$.

(b) $X'(t) = \tan(X) - 1$.

Solution: See sketches on next page. Your sketches don't need to be this accurate.

(a) The tan function is periodic with period π and the graph shows only a few periods. The graph of tan crosses the axis at *all* multiples of π. So all of these values are equilibria. The graph also has vertical asymptotes at all odd multiples of $\frac{\pi}{2}$. Notice carefully in which direction the arrows are pointing. All of the equilibria are unstable.

(b) This is the same graph as in (a), BUT shifted down by 1. *You need to look closely to see the shift.* This means it will now cross the axis at all the points where $\tan(X) = 1$. Now $\tan\frac{\pi}{4} = 1$, so one equilibrium is $X_{eq} = \frac{\pi}{4}$. Using the periodicity, the complete set of equilibria are given by $X_{eq} = \frac{\pi}{4} + n\pi$ for all integers n.



8. Some extremely primitive aquatic organisms absorb nutrients through their skin, and the rate at which this occurs is proportional to their surface area. On the other hand, their need for nutrients is related to their volume. Thus their growth is affected by two quantities: a positive contribution from their surface area and a negative contribution from their volume. If L represents the length of such an organism then a simple model of growth might be

$$\frac{dL}{dt} = aL^2 - bL^3$$

where the first term is proportional to surface area, and the second term is proportional to volume. The two constants a and b are both positive and depend on details of the biology.

How many equilibria does the model have? Does this model have a positive equilibrium?

(Advanced Question) Is the positive equilibrium stable? [Hint: You may want to first show that the sign of $\frac{dL}{dt}$ is the same as the sign of $a - bL$.]

Solution: The equilibrium condition is

$$0 = aL_{eq}^2 - bL_{eq}^3.$$

The expression on the right can be factorised to give

$$0 = L_{eq}^2(a - bLeq).$$

There are two equilibria, one at $L_{eq} = 0$ and another one at $L_{eq} = \frac{a}{b}$. Since a and b are both positive, this second equilibrium is a positive equilibrium.

We can rewrite the DE as

$$\frac{dL}{dt} = L^2(a - bL)$$

and since L^2 is cannot be negative, then $\frac{dL}{dt}$ and $a - bL$ must have the same sign. Thus if $L > \frac{a}{b}$ the growth rate will be negative, but if $L < \frac{a}{b}$ the growth rate will be positive. Thus, the direction of change is always towards $\frac{a}{b}$ so this equilibrium is stable.