Solutions to Tutorial 3

Preparatory Questions

1. In each of the following difference equations, write down the next four terms after the given term(s). Try not to use your calculator!

(a) $u_{n+1} = 5u_n$ given $u_0 = 2$.

(b) $X_{n+1} = 3 + X_n$ with $X_0 = 3$.

(c) $P_{n+2} = 3P_{n+1} - 2P_n$ with $P_0 = 1$ and $P_1 = 2$.

Solution:

(a)

$$\begin{aligned} u_1 &= 5u_0 = 5 \times 2 = 10, \\ u_2 &= 5 \times 10 = 50, \\ u_3 &= 250, \\ u_4 &= 1250. \end{aligned}$$

(b)

$$\begin{aligned} X_1 &= 3 + X_0 = 3 + 3 = 6, \\ X_2 &= 3 + 6 = 9, \\ X_3 &= 12, \\ X_4 &= 15. \end{aligned}$$

(c)

$$\begin{aligned} P_2 &= 3P_1 - 2P_0 = 3 \times 2 - 2 \times 1 = 4, \\ P_3 &= 3P_2 - 2P_1 = 3 \times 4 - 2 \times 2 = 8, \\ P_4 &= 16, \\ P_5 &= 32. \end{aligned}$$

2. First write down the general solution to each of the following difference equations. (You may need to rearrange the equation first to get it into a form you recognise. Pay careful attention to the subscripts, especially which subscript is larger and which subscript is smaller.) Then calculate the particular solution that satisfies the given condition.

(a) $3P_{m+1} = P_m$ with $P_0 = 2$.

(b) $3P_m = P_{m-1}$ with $P_0 = 2$.

(c) $3Q_{n+1} - 3Q_n + 4 = 0$ with $Q_0 + Q_3 = 10$.

(d) $4X_{n-1} + 3X_n = 0$ with $X_1 = 1$.

(e) $aY_{n+1} - bY_n = 0$ with $Y_3 = 4$.

(f) $A_m - A_{m+1} = d - c$ with $A_1 = c$.

Solution: *Remember that you can use any letter you like (uppercase or lowercase) for the arbitrary constant provided it has not been used for something else.*

(a) Rearrange first $P_{m+1} = \frac{1}{3}P_m$. General solution is $P_m = A\left(\frac{1}{3}\right)^m$ where A is arbitrary. Thus $P_0 = A = 2$ and the particular solution is $P_m = 2\left(\frac{1}{3}\right)^m$.

(b) This is exactly the same as the previous part.

(c) Rearrange first $Q_{n+1} = Q_n - \frac{4}{3}$. General solution is $Q_n = -\frac{4}{3}n + C$ where C is arbitrary. Thus $Q_0 = C$ and $Q_3 = C - 4$ and, so $Q_0 + Q_3 = 2C - 4 = 10$. Hence $C = 7$ and the particular solution is $Q_n = 7 - \frac{4}{3}n$.

(d) Rearrange first $X_n = -\frac{4}{3}X_{n-1}$. Notice that the *greater* subscript is on the term on the left of the equals sign and that if you feel more comfortable rewriting the recurrence relation as $X_{n+1} = -\frac{4}{3}X_n$ then that is okay. General solution is $X_n = A\left(-\frac{4}{3}\right)^n$ where A is arbitrary. Now $X_1 = A(-\frac{4}{3}) = 1$. Thus, $A = -\frac{3}{4}$ and the particular solution is $X_n = -\frac{3}{4}(-\frac{4}{3})^n$.

(e) Rearrange first $Y_{n+1} = \frac{b}{a}Y_n$. General solution is $Y_n = A\left(\frac{b}{a}\right)^n$ where A is arbitrary. Thus $Y_3 = A(\frac{b}{a})^3 = 4$. Thus, $A = 4(\frac{b}{a})^{-3} = 4(\frac{a}{b})^3$ and the particular solution is $Y_n = 4(\frac{b}{a})^{-3}(\frac{b}{a})^n = 4(\frac{b}{a})^{n-3}$.

(f) Rearrange first to give $A_{m+1} - A_m = c - d$. General solution is $A_m = (c-d)m + C$ where C is arbitrary. Now $A_1 = (c-d) + C = c$ thus $C = d$. The particular solution is $A_m = (c-d)m + d$. You can write the solution as $A_n = (c-d)n + d$ if you like.

3. Use your favourite search engine to investigate the models associated with the names below. For each model try to find out what field it is used in, how many difference equations are used in the model and what the order is of each equation.

(a) Nicholson-Bailey

(b) Beverton-Holt

(c) Ricker

(d) Gottman, Swanson & Murray

(e) Logistic map

(f) Arnold's discrete cat map

Solution:

(a) The Nicholson-Bailey model models host-parasite interaction using two difference equations each of which is first-order.

(b) The BevertonHolt model was first used to model fisheries in 1957 and uses a single difference equation which is first-order.

(c) The Ricker model is an alternative model to Beverton-Holt and also uses a single difference equation which is first-order.

(d) Gottman, Swanson & Murray created this model to describe marital conflicts. It uses two difference equations (one for the husband and one for the wife) and each equatuion is first order. It was first published in the Journal of Family Psychology in 1999. A more recent a free version can be found here http://psr.sagepub.com/cgi/reprint/6/4/326 or for those who know what a digital-object-identifier is you can find the paper using DOI: 10.1207/S15327957PSPR0604_07

(e) The Logistic map is a famous equation used to study chaos. It is a single difference equation of first-order.

(f) Arnold's discrete cat map is a pair of first-order difference equations defined using modulo arithmetic (similar to what is taught in MATH1014) that was invented to study chaos.

4. (a) Show that if $X_n = n^2$ that $X_{n+1} - X_{n-1} = 4n$.

(b) Substitute $X_n = n^2$ into the expression $X_{n+2} - 2X_{n+1} + X_n$ and simplify as much as possible.

(c) Substitute $X_n = n(n+1)$ into the expression $X_{n+2} - 2X_{n+1} + X_n$ and simplify as much as possible.

(d) Substitute $X_n = n\,2^n$ into the expression $X_{n+1} - 4X_n + 4X_{n-1}$ and simplify as much as possible.

Solution:

(a)

$$X_{n+1} - X_{n-1} = (n+1)^2 - (n-1)^2 = (n^2 + 2n + 1) - (n^2 - 2n + 1) = 4n.$$

(b)

$$X_{n+2} - 2X_{n+1} + X_n = (n+2)^2 - 2(n+1)^2 + n^2 = (n^2+4n+4) - 2(n^2+2n+1) + n^2 = 2.$$

(c) Now if $X_n = n(n+1)$, then $X_{n+1} = (n+1)(n+2)$ and $X_{n+2} = (n+2)(n+3)$ thus

$$\begin{aligned} X_{n+2} - 2X_{n+1} + X_n &= (n+2)(n+3) - 2(n+1)(n+2) + n(n+1) \\ &= (n^2 + 5n + 6) - 2(n^2 + 3n + 2) + (n^2 + n) \\ &= 2. \end{aligned}$$

(d) Now if $X_n = n\,2^n$ then $X_{n-1} = (n-1)2^{n-1}$ and $X_{n+1} = (n+1)2^{n+1}$ thus

$$X_{n+1} - 4X_n + 4X_{n-1} = (n+1)2^{n+1} - 4n\,2^n + 4(n-1)2^{n-1}.$$

Pull a common factor of 2^{n-1} out the front this gives

$$X_{n+1} - 4X_n + 4X_{n-1} = 2^{n-1}[(n+1)2^2 - 4n\,2 + 4(n-1)] = 2^{n-1}[4(n+1) - 8n + 4(n-1)] = 0.$$

Tutorial Questions

5. Wildlife Service records show that the number of female kangaroos in Ecopark is increasing at the rate of 5% each year. At the beginning of 1997 there were 120 females in the park. Let P_n be the number of females in the mob at the beginning of the nth year from 1997. Thus $P_0 = 120$.

(a) Write down a recurrence relation representing this model.

(b) Staring with P_0 and using a calculator, use the recurrence relation to find the anticipated number of females in the mob in each of the next 3 years.

(c) Write down the general solution to the difference equation and the particular solution that satisfies $P_0 = 120$.

(d) Determine how long it would be before the number of females in the mob doubles.

Solution:

(a) A suitable recurrence relation is $P_{n+1} = 1.05P_n$. You could also write this in the form

$$\frac{P_{n+1} - P_n}{P_n} = \frac{5}{100}.$$

(b) Now $P_1 = 1.05P_0 = 1.05 \times 120 = 126$; $P_2 = 1.05 \times 126 = 132.3$ and $P_3 = 1.05 \times 132.3 = 138.9$. Thus the anticipated number of females at the beginning of 1998 is 126; the anticipated number of females at the beginning of 1999 is 132 and the anticipated number of females at the beginning of 2000 is 139.

(c) The general solution is $P_n = A\,(1.05)^n$. The particular solution is $P_n = 120\,(1.05)^n$.

(d) We seek the smallest integer n so that $P_n > 2P_0$. Thus we need to solve the inequality

$$120(1.05)^n > 2 \times 120$$

We can cancel the 120 from both sides and then take logarithms. [Note: Since the logarithm function is an increasing function, using it does not affect the direction of the inequality!] Thus,

$$n \ln(1.05) > \ln(2)$$

next, divide both sides by ln(1.05). [Note: Since ln(1.05) is positive, dividing by it does not affect the direction of the inequality!] Thus,

$$n > \frac{\ln(2)}{\ln(1.05)} \approx 14.206$$

Thus we expect P_{15} to be the first term which exceeds twice the original number. In fact $P_{14} \approx 238 < 2 \times 120$ and $P_{15} \approx 249 > 2 \times 120$

Hence, at the beginning of the year 2011 we would expect nearly 240 females in the park, but by the beginning of the next year, 2012, we would expect there to be slightly more than double the original number of females.

6. The average weight of baby pigs, recorded in a piggery, increases by 2.3% per week.

(a) Write down a difference equation for w_n, the average weight after n weeks.

(b) The initial average weight is $w_0 = 10.5$ kg. For $n = 1, \ldots, 5$ use a calculator to find w_n.

(c) After how many weeks will the average weight of the pigs exceed 13.5 kg?

Solution:

(a) Since the average weight increase in a week is 2.3%, we have

$$w_{n+1} - w_n \text{ (the weight increase) } = \frac{2.3}{100} w_n$$

which can be rewritten

$$w_{n+1} = w_n + \frac{2.3}{100} w_n = 1.023 w_n.$$

(b) $w_0 = 10.5$
$w_1 = 1.023w_0 = 1.023 \times 10.5 = 10.7415$
$w_2 = 1.023w_1 = (1.023)^2 \times 10.5 \approx 10.989$
$w_3 = 1.023w_2 = (1.023)^3 \times 10.5 \approx 11.241$
$w_4 = 1.023w_3 = (1.023)^4 \times 10.5 \approx 11.500$
$w_5 = 1.023w_4 = (1.023)^5 \times 10.5 \approx 11.764$

(c) To determine after how many weeks the average weight will exceed 13.5kg, we need to find the smallest integer n for which

$$10.5(1.023)^n > 13.5$$

i.e.,

$$(1.023)^n > \frac{13.5}{10.5}$$

i.e., (taking logs)

$$n \ln(1.023) > \ln\left(\frac{13.5}{10.5}\right)$$

or

$$n > \frac{\ln\left(\frac{13.5}{10.5}\right)}{\ln(1.023)} = 11.05\cdots.$$

So after 12 weeks the average weight exceeds 13.5kg. After 11 weeks the average weight is only 13.48 kg but after 12 weeks it is 13.79 kg.

7. Facebook is a social networking website that opened to the public in 2006. comScore is a marketing research company that monitors the popularity of many internet activities by keeping track on a monthly basis of how often the website is used and by how many people. Let F_n represent the number of unique visitors to the Facebook website during month n. According to comScore 19,000 people visited Facebook in December 2006. Let's call that month zero, thus $F_0 = 19,000$. A year later in December 2007 that number had grown to 34,600. Thus $F_{12} = 34,600$.

(a) Assume the growth in the popularity of Facebook can be modelled as a geometric progression using

$$F_{n+1} = rF_n.$$

Use the given information to calculate the value of r to four decimal places. Then, reinterpret your answer as a percentage growth per month.

(b) Which month corresponds to F_{20}? What value does this model predict for F_{20} (to the nearest hundred)?

(c) Which term in the sequence would represent the number of visitors to Facebook in August 2009? What does this model predict for the number of visitors in that month (to the nearest thousand)? *If you have spare time, you can check out how bad this model actually is by looking up current usage trends online.*

Solution:

(a) The general solution is $F_n = Ar^n$. The first piece of information is that $F_0 = 19,000$, thus this means $A = 19,000$. The second piece of information is that $F_{12} = 34,600$ this gives the equation

$$34,600 = 19,000 \times r^{12}.$$

We can rearrange this equation for r to give

$$r = (\frac{34,600}{19,000})^{\frac{1}{12}} \approx 1.0512$$

This is slightly more than 5% growth per month.

(b) F_{20} corresponds to August 2008. The model predicts that $F_{20} \approx 19,000(1.0512)^{20} \approx 51,600$.

(c) The month of August 2009 would correspond to F_{32}. The model predicts $F_{32} \approx 19,000(1.0512)^{32} \approx 94,000$. *By the way, the actual number, this month, will be well over 100 million which demonstrates that social networking continues to grow much, much faster than a geometric progression would predict. Recently, in just one month, visitors increased by over 150% rather than the 5% predicted by this model, but other sites are going out of fashion: MySpace is currently only increasing at about 3% per month.*

8. Use the information from Question 4 to answer the following questions.

(a) Find a particular solution to $X_{n+1} - X_{n-1} = n$.

(b) Find two different particular solutions to $X_{n+2} - 2X_{n+1} + X_n = 1$.

(c) Find a particular solution to $X_{n+1} - 4X_n + 4X_{n-1} = 0$. [Advanced: Try to guess some more solutions.]

Solution:

(a) Since $X_n = n^2$ satisfies $X_{n+1} - X_{n-1} = 4n$ then a particular solution to $X_{n+1} - X_{n-1} = n$ will be $X_n = \frac{1}{4}n^2$ which you can check by substitution.

(b) Since both $X_n = n^2$ and $X_n = n(n+1)$ satisfy $X_{n+2} - 2X_{n+1} + X_n = 2$ then particular solutions to $X_{n+2} - 2X_{n+1} + X_n = 1$ are $X_n = \frac{1}{2}n^2$ and $X_n = \frac{1}{2}n(n+1)$ which you can check by substitution.

(c) Since $X_n = n\,2^n$ satisfies $X_{n+1} - 4X_n + 4X_{n-1} = 0$ it is a particular solution. [Advanced: If you multiply this solution by an arbitrary constant it still satisfies the equation so $X_n = A\,n\,2^n$ are also particular solutions for all values of A. This is still not the general solution, because the general solution would require two arbitrary constants.]