Tutorial 4

Preparatory Questions

1. Find the equilibrium (fixed-point) solutions to each of the following recurrence relations.

(a) $P_{n+1} = 5P_n - 3$.

(b) $P_{n+1} - P_n = kP_n - d$.

(c) $3P_{n+1} + 2P_n + 5 = 0$.

(d) $7Q_{n+1} + Q_n^2 - 4Q_n = 10$.

(e) $2Y_{n+2} + 3Y_{n+1}^2 + 2Y_n + 1 = 0$.

(f) $aY_{n+2} + bY_{n+1}^2 + cY_n + d = 0$.

[Be careful when using the quadratic formula to find roots of quadratics.]

2. Each of the following difference equations has the form $X_{n+1} = F(X_n)$ or can be rearranged so that it is in this form.

In each case, calculate the equilibrium values. Then rearrange the equation until it is in the form $X_{n+1} = F(X_n)$, and determine stability using the magnitude of $F'(X_{eq})$ for each equilibrium solution.

(a) $X_{n+1} = 3 - 2X_n$.

(b) $4X_{n+1} = 2 - 3X_n$.

(c) $X_{n+1} + 4X_n = X_n^2 + 6$.

(d) $X_{n+1} = e^{X_n} + X_n$.

(e) $X_{n+1} - X_n = e^{X_n} - 3$.

Tutorial Questions

3. The records kept by a bookseller when marketing a popular magazine suggest that only 60% of her customers renew their subscription each year, an additional 70 new customers take out a subscription each year. At the beginning of 2009 she had 400 customers.

Let C_n be the number of customers at the beginning of the nth year after 2009. Thus, $C_0 = 400$.

A year later 60% of these customers (which is 240) renew their subscription and there are 70 new customers. Thus $C_1 = 240 + 70 = 310$.

(a) Write down a difference equation using C_{n+1} and C_n which models this behaviour.

(b) Determine the anticipated number of customers at the beginning of each of 2011, 2012 and 2013.

(c) Does this model have an equilibrium?

(d) Determine the stability of any equilibrium solutions.

(e) Predict the long term behaviour of the magazine subscriptions according to this model.

4. Let W_n be the population of cane-toads in a certain part of Western Australia in year n. Consider the following model

$$W_{n+1} = a + rW_n - eW_n$$

where $r > 1$, $a > 0$ and $e > 0$.

The first term a represents the migration of cane-toads across the country and is assumed to be a constant amount each year. The second term represents the natural reproduction of the cane toads. The last term represents the effort to eradicate cane-toads. Since there are very few canetoads in WA, they are hard to find and the number which are eradicated each year is proportional to their population.

(a) Does this model have an equilibrium population?

(b) Under what condition is the equilibrium positive?

(c) Under what condition is the equilibrium stable?

(d) Use your calculator to study the population over 10 years starting with $W_0 = 0$, $a = 1000$, $r = 1.1$ and $e = 0.5$. What is the long term prediction for this population?

(e) If $r = 1.1$ what is the smallest value of the effort e which will keep the cane-toad population stable?

5. Use a sketch to help find any equilibrium solutions (fixed-points) of the equation

$$P_{n+1} = P_n^3 + 3P_n - 5.$$

Are the equilibrium solutions stable or unstable?

[Hints: The derivative of a cubic function is a quadratic function. The square of a number cannot be negative!]

6. Consider the recurrence relation

$$Q_{n+1} = \sin(Q_n) + 1.$$

(a) Starting with $Q_0 = \frac{\pi}{2}$ and calculating in radians, use a calculator to work out the first 6 terms in the sequence. (Do the calculation to the full accuracy of your calculator but you may write down the numbers to 3 decimal places to save space.)

(b) How many equilibrium solutions (fixed-points) does equation have?

A sketch that was used in Tutorial 2 is useful for solving this problem.

(c) Determine the stability of any equilibrium solutions you find.

[Hint: The magnitude of the cosine function is always less than or equal to one and the only solutions of $|\cos(x)| = 1$ are the integer multiples of π.]

Partial solutions and/or hints to some of the preparatory questions:

1(a) one equilibrium at $\frac{3}{4}$

1(b) and 1(c) have one equilibrium each

1(d) two equilibria at 2 and -5

1(e) has two equilibria

2(b) one equilibrium at $\frac{2}{7}$, stable

2(c) two equilibria at 2 and 3: one of them is stable the other is not