Solutions to Tutorial 4

Preparatory Questions

1. Find the equilibrium (fixed-point) solutions to each of the following recurrence relations.

(a) $P_{n+1} = 5P_n - 3$.

(b) $P_{n+1} - P_n = kP_n - d$.

(c) $3P_{n+1} + 2P_n + 5 = 0$.

(d) $7Q_{n+1} + Q_n^2 - 4Q_n = 10$.

(e) $2Y_{n+2} + 3Y_{n+1}^2 + 2Y_n + 1 = 0$.

(f) $aY_{n+2} + bY_{n+1}^2 + cY_n + d = 0$.

[Be careful when using the quadratic formula to find roots of quadratics.]

Solution:

(a) The equilibrium condition is $P_{eq} = 5P_{eq} - 3$. There is only one solution $P_{eq} = \frac{3}{4}$.

(b) The equilibrium condition is $0 = kP_{eq} - d$. There is only one solution $P_{eq} = \frac{d}{k}$.

(c) The equilibrium condition is $3P_{eq}+2P_{eq}+5 = 0$. There is only one solution $P_{eq} = -1$.

(d) The equilibrium condition is $7Q_{eq} + Q_{eq}^2 - 4Q_{eq} = 10$. This is a quadratic equation and can be rearranged to give $Q_{eq}^2 + 3Q_{eq} - 10 = 0$ it has two solutions which are $Q_{eq} = 2$ and $Q_{eq} = -5$.

(e) The equilibrium condition is $2Y_{eq} + 3Y_{eq}^2 + 2Y_{eq} + 1 = 0$. This is a quadratic equation and can be rearranged to give $3Y_{eq}^2+4Y_{eq}+1 = 0$. It can be solved using the quadratic formula and the solutions are

$$Y_{eq} = \frac{-4 \pm \sqrt{4^2 - 12}}{6} = \frac{-4 \pm 2}{6}.$$

The solutions are thus -1 or $-\frac{1}{3}$. The quadratic can also be factorised $3Y_{eq}^2+4Y_{eq}+1 = (Y_{eq} + 1)(3Y_{eq} + 1) = 0$

(f) The equilibrium condition is $aY_{eq} + bY_{eq}^2 + cY_{eq} + d = 0$. This is a quadratic equation and can be rearranged to give $bY_{eq}^2 + (a + c)Y_{eq} + d = 0$. It can be solved using the quadratic formula and the solutions are

$$Y_{eq} = \frac{-(a + c) \pm \sqrt{(a + c)^2 - 4bd}}{2b}.$$

2. Each of the following difference equations has the form $X_{n+1} = F(X_n)$ or can be rearranged so that it is in this form.

In each case, calculate the equilibrium values. Then rearrange the equation until it is in the form $X_{n+1} = F(X_n)$, and determine stability using the magnitude of $F'(X_{eq})$ for each equilibrium solution.

(a) $X_{n+1} = 3 - 2X_n$.

(b) $4X_{n+1} = 2 - 3X_n$.

(c) $X_{n+1} + 4X_n = X_n^2 + 6$.

(d) $X_{n+1} = e^{X_n} + X_n$.

(e) $X_{n+1} - X_n = e^{X_n} - 3$.

Solution:

(a) The equilibrium condition is $X_{eq} = 3 - 2X_{eq}$ which has only one solution $X_{eq} = 1$. For this model $F(x) = 3 - 2x$ and thus $F'(x) = -2$. Since the absolute value of the slope is greater than 1, the equilibrium is unstable.

(b) The equilibrium condition is $4X_{eq} = 2 - 3X_{eq}$ which has only one solution $X_{eq} = \frac{2}{7}$. After rearranging, the equation is $X_{n+1} = \frac{1}{2} - \frac{3}{4}X_n$. Thus, for this model $F(x) = \frac{1}{2} - \frac{3}{4}x$ and thus $F'(x) = -\frac{3}{4}$. Since the absolute value of the slope is less than 1, the equilibrium is stable.

(c) The equilibrium condition is $X_{eq}+4X_{eq} = X_{eq}^2+6$ which is the same as $X_{eq}^2-5X_{eq}+6 = 0$. This quadratic has two solutions $X_{eq} = 2$ and $X_{eq} = 3$.

After rearranging, the equation is $X_{n+1} = X_n^2 - 4X_n + 6$. Thus, for this model $F(x) = x^2 - 4x + 6$ and thus $F'(x) = 2x - 4$. At the equilibrium $X_{eq} = 2$ the slope is $F'(2) = 2 \times 2 - 4 = 0$ since this is less than one in magnitude, this equilibrium is stable. At the equilibrium $X_{eq} = 3$ the slope is $F'(3) = 2 \times 3 - 4 = 2$ since this is greater than one in magnitude, this equilibrium is unstable.

(d) The equilibrium condition is $X_{eq} = e^{X_{eq}} + X_{eq}$ which is the same as $e^{X_{eq}} = 0$ which has no solutions.

(e) The equilibrium condition is $X_{eq} - X_{eq} = e^{X_{eq}} - 3$ which is the same as $e^{X_{eq}} = 3$ which has one solution $X_{eq} = \ln(3)$.

After rearranging, the equation is $X_{n+1} = e^{X_n} + X_n - 3$. Thus, for this model $F(x) = e^x + x - 3$ and thus $F'(x) = e^x + 1$. Thus at the equilibrium $F'(\ln(3)) = e^{\ln(3)} + 1 = 3 + 1 = 4$. Since this is greater than one in magnitude, this equilibrium is unstable.

Tutorial Questions

3. The records kept by a bookseller when marketing a popular magazine suggest that only 60% of her customers renew their subscription each year, an additional 70 new customers take out a subscription each year. At the beginning of 2009 she had 400 customers.

Let C_n be the number of customers at the beginning of the nth year after 2009. Thus, $C_0 = 400$.

A year later 60% of these customers (which is 240) renew their subscription and there are 70 new customers. Thus $C_1 = 240 + 70 = 310$.

(a) Write down a difference equation using C_{n+1} and C_n which models this behaviour.

(b) Determine the anticipated number of customers at the beginning of each of 2011, 2012 and 2013.

(c) Does this model have an equilibrium?

(d) Determine the stability of any equilibrium solutions.

(e) Predict the long term behaviour of the magazine subscriptions according to this model.

Solution:

(a) The number of customers renewing in after $n+1$ years is $0.6C_n$ and the number of new customers is 70. Thus the total number of customers after $n+1$ years is

$$C_{n+1} = 0.6C_n + 70.$$

(b) The beginning of of 2011 corresponds to C_2. Thus we have

$$C_2 = 0.6C_1 + 70 = 0.6 \times 310 + 70 = 256.$$

The beginning of of 2012 corresponds to C_3. Thus we have

$$C_3 = 0.6C_2 + 70 = 0.6 \times 256 + 70 = 223.6$$

The beginning of of 2013 corresponds to C_4. Thus we have

$$C_4 = 0.6C_3 + 70 = 0.6 \times 223.6 + 70 = 204.16$$

Obviously the model is only an approximation since you can't have a fractional number of customers.

(c) The equilibrium condition is $C_{eq} = 0.6C_{eq} + 70$ which is the same as $0.4C_{eq} = 70$ and thus there is one equilibrium $C_{eq} = 175$.

(d) If we write the model in the form $C_{n+1} = F(C_n)$ then $F(x) = 0.6x + 70$ and $F'(x) = 0.6$ since the magnitude of this slope is less than one the equilibrium is stable.

(e) In the long term, the number of subscriptions will stabilise to 175 customers.

4. Let W_n be the population of cane-toads in a certain part of Western Australia in year n. Consider the following model

$$W_{n+1} = a + rW_n - eW_n$$

where $r > 1$, $a > 0$ and $e > 0$.

The first term a represents the migration of cane-toads across the country and is assumed to be a constant amount each year. The second term represents the natural reproduction of the cane toads. The last term represents the effort to eradicate cane-toads. Since there are very few canetoads in WA, they are hard to find and the number which are eradicated each year is proportional to their population.

(a) Does this model have an equilibrium population?

(b) Under what condition is the equilibrium positive?

(c) Under what condition is the equilibrium stable?

(d) Use your calculator to study the population over 10 years starting with $W_0 = 0$, $a = 1000$, $r = 1.1$ and $e = 0.5$. What is the long term prediction for this population?

(e) If $r = 1.1$ what is the smallest value of the effort e which will keep the cane-toad population stable?

Solution:

(a) The equilibrium condition is $W_{eq} = a + rW_{eq} - eW_{eq}$ which has a single solution

$$W_{eq} = \frac{a}{1 - r + e}$$

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(b) Since $a > 0$ then this equilibrium is positive if $1 - r + e > 0$.

(c) The model has the form $W_{n+1} = F(W_n)$ where $F(x) = a + rx - ex$. Thus the slope is $F'(x) = r - e$. Thus the equilibrium is stable whenever $|r - e| < 1$.

(d) The sequence given by these values is

$$0, 1000, 1600, 1960, 2176, 2305.6, 2383.36, 2430.016, 2458.0096, 2474.80576, 2484.883456$$

Eventually the population stabilises at 2500.

(e) If $r = 1.1$ then the condition for a stable equilibrium is $|1.1 - e| < 1$ which is the same as $0.1 < e < 2.1$. Thus, the minimum effort required corresponds to $e = 0.1$.

5. Use a sketch to help find any equilibrium solutions (fixed-points) of the equation

$$P_{n+1} = P_n^3 + 3P_n - 5.$$

Are the equilibrium solutions stable or unstable?

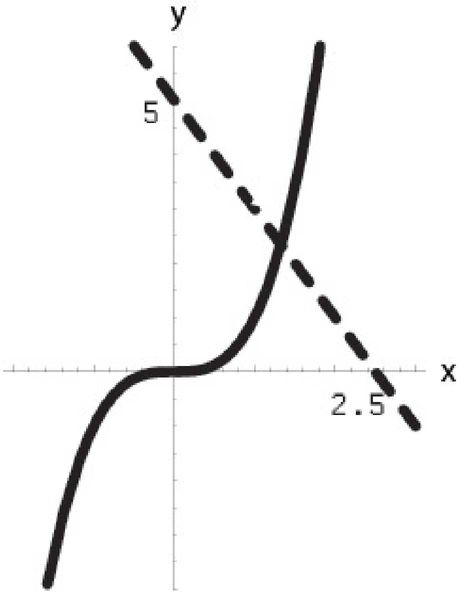
[Hints: The derivative of a cubic function is a quadratic function. The square of a number cannot be negative!]

Solution:

The equilibrium condition is $P_{eq} = P_{eq}^3 + 3P_{eq} - 5$. This is the same as

$$P_{eq}^3 = 5 - 2P_{eq}.$$

Thus, a useful diagram would be a sketch of both $y = x^3$ and $y = 5 - 2x$. We used such a diagram in Tutorial Set 2. Here is the diagram again.



This diagram reveals that there is only one equilibrium and that $0 < P_{eq} < \frac{5}{2}$.

If we write the model in the form $P_{n+1} = F(P_n)$ then $F(x) = x^3 + 3x - 5$ and the slope is $F'(x) = 3x^2 + 3$. Although we do not know exactly what the equilibrium value is, it doesn't matter, the slope is always at least 3. This is because $x^2 \geq 0$ and thus $F'(x) \geq 3$. Thus, we can deduce that the equilibrium is unstable.

6. Consider the recurrence relation

$$Q_{n+1} = \sin(Q_n) + 1.$$

(a) Starting with $Q_0 = \frac{\pi}{2}$ and calculating in radians, use a calculator to work out the first 6 terms in the sequence. (Do the calculation to the full accuracy of your calculator but you may write down the numbers to 3 decimal places to save space.)

(b) How many equilibrium solutions (fixed-points) does equation have?

A sketch that was used in Tutorial 2 is useful for solving this problem.

(c) Determine the stability of any equilibrium solutions you find.

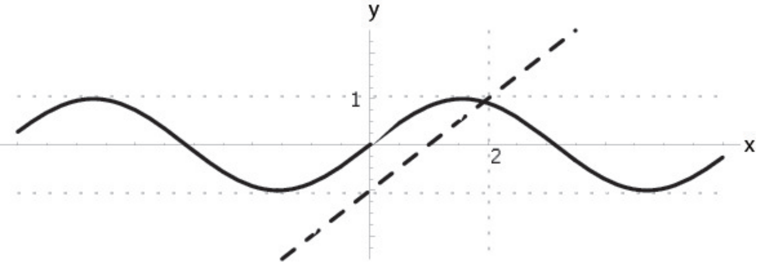
[Hint: The magnitude of the cosine function is always less than or equal to one and the only solutions of $|\cos(x)| = 1$ are the integer multiples of π.]

Solution:

(a)

$$\frac{\pi}{2} \approx 1.571, \quad 2, \quad 1.909, \quad 1.943, \quad 1.931, \quad 1.936$$

(b) The equilibrium condition is $Q_{eq} = \sin(Q_{eq}) + 1$ which is the same as $Q_{eq} - 1 = \sin(Q_{eq})$. Thus the solutions can be found visually by sketching $y = \sin(x)$ and $y = x - 1$ on the same diagram. This diagram will look like the one from Q7 in Tutorial 2. Here is the diagram again.



There is only one equilibrium solution and it is somewhere between $\frac{\pi}{2}$ and 2. The previous part of the question also helps to let us know that the equilibrium is close to 1.93

(c) If we write the model in the form $Q_{n+1} = F(Q_n)$ then $F(x) = \sin(x) + 1$ and the slope is $F'(x) = \cos(x)$. Although we do not know exactly what the equilibrium value is, it doesn't matter, the slope is less than 1 because the magnitude of cosine function is less than one *everywhere* except for exact multiples of π. But we know the equilibrium is *not* an exact multiple of π because we have an approximate idea of where it is. Thus, we can deduce that the equilibrium is stable.

The first part of the question also helps us tell that the equilibrium is stable, because the calculated sequence is getting closer and closer to a particular value.

Partial solutions and/or hints to some of the preparatory questions:

1(a) one equilibrium at $\frac{3}{4}$

1(b) and 1(c) have one equilibrium each

1(d) two equilibria at 2 and -5

1(e) has two equilibria

2(b) one equilibrium at $\frac{2}{7}$, stable

2(c) two equilibria at 2 and 3: one of them is stable the other is not