Solutions to Tutorial 5/6

Preparatory Questions

1. Give the general solution to each of the following equations.

(a) $P'(t) = 5P - 3$.

(b) $3P' + 2P + 5 = 0$.

(c) $P_{n+1} = 5P_n - 3$.

(d) $3P_{n+1} + 2P_n + 5 = 0$.

Solution:

(a) The equilibrium is $P_{eq} = \frac{3}{5}$. Thus $P(t) = \frac{3}{5} + Ae^{5t}$.

(b) The equilibrium is $P_{eq} = -\frac{5}{2}$. Thus $P(t) = -\frac{5}{2} + Ae^{-\frac{2}{3}t}$.

(c) The equilibrium is $P_{eq} = \frac{3}{4}$. Thus $P_n = \frac{3}{4} + A5^n$.

(d) The equilibrium is $P_{eq} = -1$. Thus $P_n = -1 + A(-\frac{2}{3})^n$.

2. Give the particular solution to each of the following equations.

(a) $P'(t) = 7P - 4$ with $P(0) = 1$.

(b) $2P' + 3P + 4 = 0$ with $P(0) = 0$.

(c) $P_{n+1} = 3P_n - 5$ with $P_0 = 2$.

(d) $2P_{n+1} + 5P_n - 14 = 0$ with $P_0 = 9$.

Solution:

(a) The general solution is $P(t) = \frac{4}{7} + Ae^{7t}$. Now $1 = P(0) = \frac{4}{7} + A$, thus $A = \frac{3}{7}$. Thus the particular solution is $P(t) = \frac{4}{7} + \frac{3}{7}e^{7t}$

(b) The general solution is $P(t) = -\frac{4}{3} + Ae^{-\frac{3}{2}t}$. Now $0 = P(0) = -\frac{4}{3} + A$, thus $A = \frac{4}{3}$. Thus the particular solution is $P(t) = -\frac{4}{3} + \frac{4}{3}e^{-\frac{3}{2}t}$

(c) The general solution is $P_n = \frac{5}{2} + A3^n$. Now $2 = P(0) = \frac{5}{2} + A$, thus $A = -\frac{1}{2}$. Thus the particular solution is $P_n = \frac{5}{2} - \frac{1}{2}3^n$.

(d) The general solution is $P_n = 2 + A(-\frac{5}{2})^n$. Now $9 = P(0) = 2 + A$, thus $A = 7$. Thus the particular solution is $P_n = 2 + 7(-\frac{5}{2})^n$.

3. Find the values of a and b for each of the following partial fraction expansions

(a) $\dfrac{1}{(x-3)(x-5)} = \dfrac{a}{x-3} + \dfrac{b}{x-5}$.

(b) $\dfrac{1}{(x-4)(x+7)} = \dfrac{a}{x-4} + \dfrac{b}{x+7}$.

(c) $\dfrac{1}{x^2+5x+6} = \dfrac{a}{x+2} + \dfrac{b}{x+3}$.

Solution:

(a) Multiply through by the denominator to get $1 = a(x-5) + b(x-3)$.

Method 1: Try two different values of x. If $x = 3$ then $1 = -2a$ thus $a = -\frac{1}{2}$. If $x = 5$ then $1 = 2b$ thus $b = \frac{1}{2}$.

Method 2: Rewrite as $1 = (a+b)x - 5a - 3b$. Equate coefficients, thus $a+b = 0$ and $1 = -5a - 3b$. Solve these simultaneously to get is $a = -\frac{1}{2}$ and $b = \frac{1}{2}$.

(b) $1 = a(x+7) + b(x-4)$. Thus $a = \frac{1}{11}$ and $b = -\frac{1}{11}$.

(c) $1 = a(x+3) + b(x+2)$. Thus $a = 1$ and $b = -1$.

Tutorial Questions

4. You have just bought a bottle of white wine at room temperature ($20°C$). Your thermostatically controlled and extremely expensive refrigerator maintains a constant temperature of $2°C$. You place the bottle in the fridge and 10 minutes later you check and find that the bottle is now $18°C$. Assume Newton's Law of Cooling

$$T(t) = T_{eq} + A\,e^{-kt}.$$

for the temperature (in degrees Celsius) after t minutes.

(a) Calculate T_{eq}, A and k.

(b) How long do you have to wait for the wine to be at $10°C$?

(c) How does this analysis compare with the information at `http://www.wineinfonet.com/wine-serving-temperature.html`

Solution:

(a) From the description of the problem $T_{eq} = 2$. Now $20 = T(0) = 2 + A$, thus $A = 18$. Now $18 = T(10) = 2 + 18e^{-10k}$.

Thus $\frac{16}{18} = e^{-10k}$ and taking natural logs of both side $\ln(\frac{16}{18}) = -10k$. Thus $k \approx 0.0118$.

(b) Solve the equation $10 = 2 + 18e^{-kt}$ for t. Thus $\frac{8}{18} = e^{-kt}$ and taking logs $\ln(\frac{8}{18}) = -kt$. Thus $t = 68.8$. Thus you need to wait a bit over an hour.

(c) This website recommends 10 mins for every $2°C$ drop in temperature. Thus it predicts that it will take 50 mins for the wine to drop from $20°C$ to $10°C$.

5. The annual membership of club satisfies $M_{n+1} = 0.9M_n + 10$.

(a) Give the general solution.

(b) Give the particular solution satisfying $M_0 = 5$.

(c) When does M_n first exceed 99?

Solution:

(a) The equilibrium value is 100. Thus, the general solution is $M_n = 100 + A(0.9)^n$.

(b) $5 = M_0 = 100 + A$. Thus $A = -95$. The particular solution is

$$M_n = 100 - 95(0.9)^n$$

(c)

$$99 = M_n = 100 - 95(0.9)^n$$

Rearranging this give

$$\frac{1}{95} = (0.9)^n$$

Which gives $n = -\frac{\ln(95)}{\ln(0.9)} \approx 43.22$. However, n has to be an integer. Thus the M_{44} is the first time M_n exceeds 99. One can check that $M_{43} = 100 - 95(0.9)^{43} \approx 98.976$ and $M_{44} = 100 - 95(0.9)^{44} \approx 99.079$.

6. Calculate

$$\int \frac{5}{2x^2 + 7x + 3} dx$$

using partial fractions.

Solution: First factorise the quadratic $2x^2 + 7x + 3 = (2x + 1)(x + 3)$.

Next, write

$$\frac{5}{(2x + 1)(x + 3)} = \frac{a}{2x + 1} + \frac{b}{x + 3}$$

Multiply both sides by the denominator and expand

$$5 = a(x + 3) + b(2x + 1)$$

If $x = -3$ then $5 = -5b$ thus $b = -1$.

If $x = -\frac{1}{2}$ then $5 = \frac{5}{2}a$ thus $a = 2$.

Thus

$$\int \frac{5}{2x^2 + 7x + 3} dx = \int \frac{2}{2x + 1} dx - \int \frac{1}{x + 3} dx$$

Thus the integral is

$$\ln|2x + 1| - \ln|x + 3| + C$$

which can also be rewritten

$$\ln|\frac{2x + 1}{x + 3}| + C$$

7. An organism has a relative growth rate that is inversely proportional to its age. Assume this can be modelled by the differential equation

$$\frac{1}{L}\frac{dL}{dt} = \frac{k}{t}$$

where L is the length of the organism, and t is the age in years.

(a) Use separation of variables to show that

$$\ln|L| = k\ln|t| + C$$

where C is the arbitrary constant of integration.

(b) Determine the arbitrary constant if $L = 5$ when $t = 1$.

(c) Thus, determine the value of k if $L = 20$ when $t = 2$.

(d) Give the formula for $L(t)$. Assume length and time are both positive.

(e) How long is the organism when $t = 3$?

Solution:

(a) Using separation of variables

$$\int \frac{1}{L}dL = \int \frac{k}{t}dt$$

Doing both integrals

$$\ln|L| = k\ln|t| + C$$

where C is the arbitrary constant of integration.

(b) If $L = 5$ when $t = 1$ then

$$\ln(5) = k\ln(1) + C$$

Thus $C = \ln(5)$.

(c) If $L = 20$ when $t = 2$ then

$$\ln(20) = k\ln(2) + \ln(5)$$

Thus

$$k = \frac{\ln(20) - \ln(5)}{\ln(2)} = 2$$

(d) Thus

$$\ln|L| = 2\ln|t| + \ln(5)$$

Taking exponentials of both sides

$$|L| = e^{2\ln|t| + \ln(5)} = e^{2\ln|t|}e^{\ln(5)} = 5|t|^2$$

Assuming both length and time are positive, then this can be written as

$$L(t) = 5t^2$$

.

(e) When $t = 3$ the length is $5 \times 3^2 = 45$.

Partial solutions and/or hints to some of the preparatory questions:

1(b) $P(t) = -\frac{5}{2} + Ae^{-\frac{2}{3}t}$
1(d) $P_n = -1 + A(-\frac{2}{3})^n$
2(b) $P(t) = -\frac{4}{3} + \frac{4}{3}e^{-\frac{3}{2}t}$
2(d) $P_n = 2 + 7(-\frac{5}{2})^n$.
3(c) $a = 1$ and $b = -1$.