Solutions to Tutorial 7

Preparatory Questions

1. Use separation of variables to find the general solution to each of the following equations. Rearrange your solution to give an explicit formula for $P(t)$.

(a) $P'(t) = P^2 t^3$.

(b) $P'(t) = e^{-P} e^t$.

(c) $P'(t) = P^2 e^t$.

(d) $P'(t) = e^{-P} t^3$.

Solution:

(a) Separate and integrate both sides

$$\begin{aligned} \int \frac{dP}{P^2} &= \int t^3 dt \\ -\frac{1}{P} &= \frac{1}{4}t^4 + C \end{aligned}$$

In this case it is possible to rearrange and solve for P to give

$$P = -\frac{1}{\frac{1}{4}t^4 + C}.$$

(b) Separate and integrate both sides

$$\begin{aligned} \int e^P dP &= \int e^t dt \\ e^P &= e^t + C \end{aligned}$$

In this case it is possible to solve for P by taking logs of both sides

$$P = \ln(e^t + C).$$

(c) Separate and integrate both sides

$$\begin{aligned} \int \frac{dP}{P^2} &= \int e^t dt \\ -\frac{1}{P} &= e^t + C \end{aligned}$$

In this case it is possible to rearrange and solve for P to give

$$P = -\frac{1}{e^t + C}.$$

(d) Separate and integrate both sides

$$\begin{aligned} \int e^P dP &= \int t^3 dt \\ e^P &= \frac{1}{4}t^4 + C \end{aligned}$$

In this case it is possible to solve for P by taking logs of both sides

$$P = \ln(\frac{1}{4}t^4 + C).$$

2. Use separation of variables and partial fractions to find the general solution to each of the following equations.

(a) $P'(t) = (P^2 - 4)(t^2 - 4)$.

(b) $P'(t) = \dfrac{P^2 - 4}{t^2 - 4}$.

Advanced: Rearrange the solution to give an explicit formula for $P(t)$.

Solution:

(a) Separate and integrate both sides

$$\begin{aligned}
\int \frac{dP}{P^2 - 4} &= \int (t^2 - 4)dt \\
\int \frac{dP}{(P-2)(P+2)} &= \int (t^2 - 4)dt \\
\int [\frac{\frac{1}{4}}{P-2} - \frac{\frac{1}{4}}{P+2}]dP &= \frac{1}{3}t^3 - 4t + C \\
\frac{1}{4}\ln|P-2| - \frac{1}{4}\ln|P+2| &= \frac{1}{3}t^3 - 4t + C
\end{aligned}$$

In this case it is still possible to solve for P by rearranging but it is a bit complicated

$$\begin{aligned}
\frac{1}{4}\ln|\frac{P-2}{P+2}| &= \frac{1}{3}t^3 - 4t + C \\
\frac{P-2}{P+2} &= \pm\exp(\frac{4}{3}t^3 - 16t + 4C)
\end{aligned}$$

At this point it is helpful to define $A = \pm e^{4C}$. Thus

$$\begin{aligned}
\frac{P-2}{P+2} &= A\exp(\frac{4}{3}t^3 - 16t) \\
(P-2) &= (P+2)A\exp(\frac{4}{3}t^3 - 16t) \\
P[1 - A\exp(\frac{4}{3}t^3 - 16t)] &= 2 + 2A\exp(\frac{4}{3}t^3 - 16t) \\
P &= 2\frac{1 + A\exp(\frac{4}{3}t^3 - 16t)}{1 - A\exp(\frac{4}{3}t^3 - 16t)}
\end{aligned}$$

(b) Separate and integrate both sides

$$\begin{aligned}
\int \frac{dP}{P^2 - 4} &= \int \frac{dt}{t^2 - 4} \\
\int \frac{dP}{(P-2)(P+2)} &= \int \frac{dt}{(t-2)(t+2)} \\
\int [\frac{\frac{1}{4}}{P-2} - \frac{\frac{1}{4}}{P+2}]dP &= \int [\frac{\frac{1}{4}}{t-2} - \frac{\frac{1}{4}}{t+2}]dt \\
\frac{1}{4}\ln|P-2| - \frac{1}{4}\ln|P+2| &= \frac{1}{4}\ln|t-2| - \frac{1}{4}\ln|t+2| + C
\end{aligned}$$

In this case it is still possible to solve for P by rearranging but it is very complicated

$$\begin{aligned}
\ln|\frac{P-2}{P+2}| &= \ln|\frac{t-2}{t+2}| + 4C \\
\frac{P-2}{P+2} &= \pm\frac{t-2}{t+2}e^{4C}
\end{aligned}$$

At this point it is helpful to define $A = \pm e^{4C}$. Thus

$$\begin{aligned}
\frac{P-2}{P+2} &= A\frac{t-2}{t+2} \\
(P-2) &= (P+2)A\frac{t-2}{t+2} \\
P[1 - A\frac{t-2}{t+2}] &= 2 + 2A\frac{t-2}{t+2} \\
P &= 2\frac{1 + A\frac{t-2}{t+2}}{1 - A\frac{t-2}{t+2}} \\
P &= 2\frac{t+2+A(t-2)}{t+2-A(t-2)}
\end{aligned}$$

3. Calculate the following integrals:

(a) $\displaystyle\int \frac{1}{x^2+5x+6}dx.$

(b) $\displaystyle\int \frac{1}{5+9x-2x^2}dx.$

(c) $\displaystyle\int \frac{1}{(x-1)(x-2)(x-3)}dx.$

Solution:

(a) First $x^2+5x+6 = (x+2)(x+3)$. Thus,

$$\frac{1}{x^2+5x+6} = \frac{a}{x+2} + \frac{b}{x+3} = \frac{a(x+3)+b(x+2)}{x^2+5x+6}$$

Thus $a = 1$ and $b = -1$. Finally,

$$\int \frac{1}{x^2+5x+6}dx = \int \frac{dx}{x+2} - \int \frac{dx}{x+3} = \ln|x+2| - \ln|x+3| + C$$

(b) First $5+9x-2x^2 = (2x+1)(5-x)$. Thus,

$$\frac{1}{5+9x-2x^2} = \frac{a}{2x+1} + \frac{b}{5-x} = \frac{a(5-x)+b(2x+1)}{5+9x-2x^2}$$

Thus $a = \frac{2}{11}$ and $b = \frac{1}{11}$. Finally,

$$\int \frac{1}{x^2+5x+6}dx = \frac{2}{11}\int \frac{dx}{2x+1} + \frac{1}{11}\int \frac{dx}{5-x} = \frac{1}{11}\ln|2x+1| - \frac{1}{11}\ln|5-x| + C$$

(c) First,

$$\begin{aligned}
\frac{1}{(x-1)(x-2)(x-3)} &= \frac{a}{x-1} + \frac{b}{x-2} + \frac{c}{x-3} \\
&= \frac{a(x-2)(x-3)+b(x-1)(x-3)+c(x-1)(x-2)}{(x-1)(x-2)(x-3)}
\end{aligned}$$

Try $x = 1$ to discover that $1 = 2a$.
Try $x = 2$ to discover that $1 = -b$.

Try $x = 3$ to discover that $1 = 2c$.

Thus

$$\begin{aligned} \int \frac{1}{(x-1)(x-2)(x-3)} dx &= \frac{1}{2}\int \frac{dx}{x-1} - \int \frac{dx}{x-2} + \frac{1}{2}\int \frac{dx}{x-3} \\ &= \frac{1}{2}\ln|x-1| - \ln|x-2| + \frac{1}{2}\ln|x-3| + C \end{aligned}$$

Tutorial Questions

4. Use separation of variables to find the general solution to

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Calculate the value of the arbitrary constant if $y = 2$ when $x = 0$. What geometric object does the particular solution represent?

Solution: Separate and integrate

$$\begin{aligned} \int y\,dy &= -\int x\,dx \\ \frac{1}{2}y^2 &= -\frac{1}{2}x^2 + C \end{aligned}$$

If $y = 2$ when $x = 0$, then this gives $C = 2$. Rearranging this gives

$$x^2 + y^2 = 4$$

This is a circle of radius 2.

5. An organism has a relative growth rate that oscillates on a daily basis. Assume this can be modelled by the differential equation

$$\frac{1}{V}\frac{dV}{dt} = k\cos(\frac{2\pi}{24}t)$$

where V is the volume of the organism, and t is the time in hours.

(a) Obtain the general solution using separation of variables.

(b) Determine the arbitrary constant if $V = 1$ when $t = 0$.

(c) Give the formula for $V(t)$.

Solution:

(a) Using separation of variables we get

$$\int \frac{1}{V}dV = \int k\cos(\frac{2\pi}{24}t)dt$$

Doing both integrals

$$\ln|V| = k\frac{24}{2\pi}\sin(\frac{2\pi}{24}t) + C$$

where C is the arbitrary constant of integration.

(b) If $V = 1$ when $t = 0$ then

$$\ln(1) = k\frac{24}{2\pi}\sin(0) + C$$

Thus C is also zero.

(c) Thus, taking exponentials of both sides

$$V(t) = e^{k\frac{24}{2\pi}\sin(\frac{2\pi}{24}t)}.$$

6. In Tutorial 2 we looked at an organism of length L whose growth rate could be modelled as

$$\frac{dL}{dt} = aL^2 - bL^3$$

where the first term is proportional to surface area, and the second term is proportional to volume. Let $a = 1$ and $b = 1$ in the following calculations.

(a) What is the positive equilibrium L_{eq} for this model?

(b) Find values of p, q and r such that

$$\frac{1}{L^2 - L^3} = \frac{p}{L^2} + \frac{q}{L} + \frac{r}{1 - L}.$$

(c) Use separation of variables and the above formula to find the general solution of the differential equation.

(d) Assume that the initial length of the organism (i.e. when it hatches) is 5% of its equilibrium length. Use this information to give an expression for the arbitrary constant?

(e) Give an expression for the time required for the organism to reach 95% of its equilibrium length.

Solution:

(a) $0 = L_{eq}^2 - L_{eq}^3$ has two solutions: 0 and 1, but only one *positive* solution. $L_{eq} = 1$

(b) You need

$$1 = p(1 - L) + qL(1 - L) + rL^2.$$

By trying $L = 0$ and $L = 1$ you can discover that $p = 1$ and $r = 1$. This gives

$$1 = 1 - L + qL(1 - L) + L^2$$

Trying any another value of l reveals that $q = 1$ as well. Thus,

$$\frac{1}{L^2 - L^3} = \frac{1}{L^2} + \frac{1}{L} + \frac{1}{1 - L}.$$

(c) Thus

$$\begin{aligned}
\int \frac{dL}{L^2 - L^3} &= \int dt \\
\int \frac{dL}{L^2} + \int \frac{dL}{L} + \int \frac{dL}{1 - L} &= t + C \\
-\frac{1}{L} + \ln|L| - \ln|1 - L| &= t + C
\end{aligned}$$

(d) When $t = 0$, we have $L = 0.05$ thus $C = -20 + \ln(0.05) - \ln(0.95) = -20 - \ln(19) \approx -22.944$

(e) When $L = 0.95$ we have $t = -\frac{1}{0.95} + \ln(0.95) - \ln(0.05) + 20 + \ln(19) \approx 24.83$.

7. Use separation of variables to find the general solution to

$$3P' + 2P + 5 = 0.$$

How does this method of solving the DE compare to the method introduced in Chapter 5?

Solution: First rearrange $3P' = -(2P + 5)$ then separate

$$\frac{3dP}{2P+5} = -dt$$

Then integrate

$$\int \frac{3dP}{2P+5} = -\int dt$$

to give

$$\frac{3}{2}\ln|2P+5| = -t + C$$

Rearrange to give

$$\ln|2P+5| = -\frac{2}{3}t + \frac{2}{3}C$$

or

$$2P + 5 = \pm e^{-\frac{2}{3}t + \frac{2}{3}C} = \pm e^{-\frac{2}{3}t} e^{\frac{2}{3}C}$$

or

$$P = -\frac{5}{2} \pm \frac{1}{2} e^{\frac{2}{3}C} e^{-\frac{2}{3}t}$$

Let $A = \pm\frac{1}{2}e^{\frac{2}{3}C}$ be a different way to write the arbitrary constant. This then gives the same solution as before

$$P = -\frac{5}{2} + Ae^{-\frac{2}{3}t}$$

This method yields the same answer: but it takes more calculations (which means more opportunity to make mistakes), and it is less obvious this way that $-\frac{5}{2}$ is the equilibrium.

Partial solutions and/or hints to some of the preparatory questions:

1(b) $P = \ln(e^t + C)$

1(c) $P = -\frac{1}{e^t + C}$.

2(b) $\frac{1}{4}\ln|P-2| - \frac{1}{4}\ln|P+2| = \frac{1}{4}\ln|t-2| - \frac{1}{4}\ln|t+2| + C$

3(b) $\frac{1}{11}\ln|2x+1| - \frac{1}{11}\ln|5-x| + C$