# Solutions to Tutorial 9

## Preparatory Questions

**1.** Consider the logistic difference equation

$$P_{n+1} = rP_n(1 - \frac{P_n}{K})$$

with $r = 1.1$ and $K = 100$.

(a) Find all equilibria and decide which are stable.

(b) Starting with $P_0 = 50$ use your calculator to explore the sequence generated by this equation. What is the long term behaviour?

(c) Starting with $P_0 = 150$ use your calculator to explore the sequence generated by this equation. What is the long term behaviour?

***Solution:***

(a) The equilibria are $P_{eq} = 0$ and

$$P_{eq} = \frac{r-1}{r}K = \frac{100}{11} =\approx 9.09$$

If $P_{n+1} = F(P_n)$ then the derivative of $F(P)$ is $F'(P) = r - 2rP/K$. The stability condition is $|F'(P_{eq})| < 1$. For $r = 1.1$ the equilibrium at zero is unstable, but the equilibrium at $P_{eq} = \frac{100}{11}$ is stable.

(b) The sequence slowly tends towards the limiting value of $\frac{100}{11} =\approx 9.09$

(c) The sequence starts

$$150, \quad -82.5, \quad -165.62, \quad -483.9, \quad \ldots$$

and continues with negative numbers of increasing magnitude.

**2.** Repeat Q1 using $r = 0.5$.

***Solution:***

(a) The equilibria are $P_{eq} = 0$ and

$$P_{eq} = \frac{r-1}{r}K = -100$$

For $r = 0.5$ using the same slope condition from the first question, the equilibrium at zero is stable, but the equilibrium at $P_{eq} = -100$ is unstable.

(b) The sequence starts

$$50, \quad 12.5, \quad , 5.468, \ldots$$

and tends to zero.

(c) The sequence starts

$$150, \quad -37.5, \quad -25.78, \quad -16.21, \quad \ldots$$

and tends to zero from below.

 

**3.** Repeat Q1 using $r = 1.5$.

***Solution:***

(a) The equilibria are $P_{eq} = 0$ and

$$P_{eq} = \frac{r-1}{r}K = \frac{100}{3} \approx 33.3$$

For $r = 1.5$ using the same slope condition from the first question, the equilibrium at zero is unstable, but the equilibrium at $P_{eq} = \frac{100}{3}$ is stable.

(b) The sequence starts

$$50, \quad 37.5, \quad ,35.16, \quad ,34.19, \ldots$$

and tends to gradually to $\frac{100}{3} \approx 33.3$.

(c) The sequence starts

$$150, \quad -112.5, \quad -358.59, \ldots$$

and continues with negative numbers of increasing magnitude.

## Tutorial Questions

**4.** The relative change in the size of a population $P_n$ is given by

$$\frac{P_{n+1} - P_n}{P_n} = 0.6 - 0.0004P_n.$$

(a) Determine the positive equilibrium directly from the above equation.

(b) Rewrite the model in the standard form of a logistic difference equation and determine the values of $r$ and $K$.

(c) Is the positive equilibrium stable?

***Solution:***

(a) The equilibrium condition is $0 = 0.6 - 0.0004P_{eq}$. Thus, $P_{eq} = \frac{0.6}{0.0004} = 1500$.

(b) Rearranging for $P_{n+1}$ gives

$$P_{n+1} = P_n + 0.6P_n - 0.0004P_n^2 = 1.6P_n - 0.0004P_n^2.$$

Thus $r = 1.6$ and $K = \frac{1.6}{0.0004} = 4000$.

(c) The stability condition is $|r - 2| < 1$ which is true when $r = 1.6$.

**5.** Consider the simplified logistic model $X_{n+1} = r\,X_n(1 - X_n)$ with $r = \frac{19}{6}$.

(a) Find the first 10 values of $X_n$ when $X_0 = 0.53$.

(b) Verify that these values appear to be tending towards a cycle of period 2 and find the limiting values occurring in this cycle to three decimal places.

***Solution:***

(a) 0.5300, 0.7888, 0.5275, 0.7893, 0.5267
0.7894, 0.5263, 0.7895, 0.5264, 0.7895

(b) The values oscillate between approximately 0.526 and 0.790

**6.** Consider the simplified logistic model $X_{n+1} = r\, X_n(1 - X_n)$ with $r = 3.45$.

(a) Find the first 12 values of $X_n$ when $X_0 = 0.53$.

(b) Is there periodic behaviour in this case? If so, what is the length of the cycle and find the values that occur in the cycle to three decimal places.

***Solution:***

(a)

$$0.5300, \quad 0.8594, \quad 0.4169, \quad 0.8387, \quad 0.4668,$$

$$0.8587, \quad 0.4186, \quad 0.8396, \quad 0.4645, \quad 0.8582, \quad 0.4200$$

(b)

Using a calculator we might infer that the difference equation has a cycle of period 4 converging to the approximate values

$$0.446, \quad 0.853, \quad 0.434, \quad 0.847$$

# Advanced Questions

**7.** An orange grower keeps a record of the production of each tree in the orchard. The yearly production for one particular tree for the past four years was 425, 217, 417 and 234.

(a) Use the first three values in the sequence 425, 217 and 417 to estimate the parameters $r$ and $K$ in a logistic model.
[Hint: $K \approx 500.26$ and $r \approx 3.4039$.]
It may be helpful to consider the following equations and think about where they come from and why they help you to work out $r$ and $K$

$$217 = r\,425(1 - \frac{425}{K}) \qquad \text{and} \qquad 417 = r\,217(1 - \frac{217}{K}).$$

(b) Then use the logistic model to predict the fourth value. How close is the prediction to the observed value of 234?

(c) Finally, determine the long term behaviour of the model.

***Solution:***

(a) The first equation is obtained by substituting $P_0 = 425$ and $P_1 = 217$ into the logistic model for $n = 0$.
The second equation is obtained by substituting $P_1 = 217$ and $P_2 = 417$ into the logistic model for $n = 1$.
Dividing one equation by the other we are able to eliminate $r$ to get

$$\frac{217}{417} = \frac{425\,(1 - \frac{425}{K})}{217\,(1 - \frac{217}{K})}.$$

This can be rearranged to give

$$\frac{217}{417}(1 - \frac{217}{K}) = \frac{425}{217}(1 - \frac{425}{K}).$$

and then

$$\frac{425}{217}\frac{425}{K} - \frac{217}{417}\frac{217}{K} = \frac{425}{217} - \frac{217}{417}.$$

Thus

$$\frac{719.45}{K} \approx 1.438$$

and hence $K \approx 500.26$. Substitute this value of $K$ into either of the original equations to find $r$.

(b) The model predicts

$$P_3 = rP_2(1 - \frac{P_2}{K}) \approx 236.25$$

which is quite close to the observed value of 234.

(c) The model predicts a cycle of period 2 about the values $\approx 421$ and $\approx 225$.

**8.** Over a five year period a pear tree produces crops of 320, 198, 310, 216 and 308 pears.

(a) Using the same method as in the previous question and the first three values in the sequence estimate the parameters $r$ and $K$ for a logistic model.

(b) Compare the predictions of the model with the values 216 and 308 for the 4th and 5th year's production.

(c) What is the long term behaviour of this model?

***Solution:***

(a) The first equation is obtained by substituting $P_0 = 320$ and $P_1 = 198$ into the logistic model for $n = 0$.

The second equation is obtained by substituting $P_1 = 198$ and $P_2 = 310$ into the logistic model for $n = 1$.

$$198 = r\,320(1 - \frac{320}{K}) \qquad \text{and} \qquad 310 = r\,198(1 - \frac{198}{K}).$$

Dividing one equation by the other we are able to eliminate $r$ to get

$$\frac{198}{310} = \frac{320}{198}\frac{(1 - \frac{320}{K})}{(1 - \frac{198}{K})}.$$

This can be rearranged to give

$$\frac{198}{310}(1 - \frac{198}{K}) = \frac{320}{198}(1 - \frac{320}{K}).$$

and then

$$\frac{320}{198}\frac{320}{K} - \frac{198}{310}\frac{198}{K} = \frac{320}{198} - \frac{198}{310}.$$

Thus

$$\frac{390.7}{K} \approx 0.97745$$

and hence $K \approx 399.7$. Substitute this value of $K$ into either of the original equations to find $r \approx 3.103$.

(b) The model predicts

$$P_3 = rP_2(1 - \frac{P_2}{K}) \approx 215.87$$

which is quite close to the observed value of 216.

The model predicts

$$P_4 = rP_3(1 - \frac{P_3}{K}) \approx 308.07$$

which is quite close to the observed value of 308.

(c) The model predicts a cycle of period 2 about the values $\approx 306$ and $\approx 222$.