SS 2065 Assignment 1 Solutions

1. Using the result that $\frac{dx}{dt} = \mu x$ has solution $x(t) = Ae^{\mu t}$ we have

$$x(t) = Ae^{-5t}$$

$$x(0) = A = -3$$

$$\therefore \ x(t) = -3e^{-5t}$$ (2)

2. The auxiliary equation is

$$\lambda^2 - 6\lambda + 9 = 0$$

$$(\lambda - 3)^2 = 0$$

$$\therefore \lambda = 3, 3$$

$$\therefore \ y = C_1 e^{3x} + C_2 x e^{3x}$$ (2)

3. (a) The auxiliary equation is

$$\lambda^2 - 4\lambda - 3 = 0$$

$$\therefore \quad \lambda = \frac{4 \pm \sqrt{16+12}}{2} = 2 \pm \sqrt{7}$$

$$\therefore \text{(i)} \ y = C_1 e^{(2+\sqrt{7})x} + C_2 e^{(2-\sqrt{7})x}$$ (2)

$$\text{(ii)} \ y = D_1 e^{2x} \cosh(\sqrt{7}x) + D_2 e^{2x} \sinh(\sqrt{7}x)$$ (1)

[See Tut 1, Q 5].

(b) Using form (i), $y(0) = 0 \Rightarrow$

$$0 = C_1 + C_2$$

$$y'(x) = C_1(2+\sqrt{7})e^{(2+\sqrt{7})x} + C_2(2-\sqrt{7})e^{(2-\sqrt{7})x}$$

$y'(0) = 1 \quad \Rightarrow$

$$1 = C_1(2+\sqrt{7}) + C_2(2-\sqrt{7})$$

$$= 2\underbrace{(C_1+C_2)}_{=0,\text{ since } C_1+C_2=0} + \sqrt{7}(C_1-C_2)$$

$\therefore$

$$C_1 + C_2 = 0$$

$$C_1 - C_2 = \frac{1}{\sqrt{7}}.$$

Adding $\Rightarrow \quad C_1 = \frac{1}{2\sqrt{7}}$

Subtracting $\Rightarrow \quad C_2 = -\frac{1}{2\sqrt{7}}$.

$$\therefore y = \frac{1}{2\sqrt{7}} e^{2x}\left(e^{\sqrt{7}x} - e^{-\sqrt{7}x}\right). \qquad 3$$

Using $\sinh w = \frac{1}{2}(e^{w} - e^{-w})$, this can be written

$$y = \frac{1}{\sqrt{7}} e^{2x}\sinh(\sqrt{7}x)$$

<u>Alternatively</u>, using form (ii), $y(0)=0 \Rightarrow$

$$0 = D_1, \quad (\text{since } \cosh 0 = 1,\ \sinh 0 = 0)$$

$$\therefore y = D_2 e^{2x}\sinh(\sqrt{7}x)$$

$$y' = D_2\, 2e^{2x}\sinh(\sqrt{7}x) + D_2\, e^{2x}\sqrt{7}\cosh(\sqrt{7}x)$$

$$\left(\text{since } \frac{d}{dx}\sinh x = \cosh x\right).$$

$y'(0)=1 \Rightarrow \quad 1 = D_2\sqrt{7} \quad \Rightarrow \quad D_2 = \frac{1}{\sqrt{7}}$

$$\therefore y = \frac{1}{\sqrt{7}} e^{2x}\sinh(\sqrt{7}x).$$

4. The auxiliary equation is

$$\lambda^2 - 2\lambda + 5 = 0$$

$$\therefore \quad \lambda = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$$

$\therefore$ The solution of the homogeneous equation is

$$y_h = e^x(C_1 \cos 2x + C_2 \sin 2x)$$ (2)

For a particular solution, try

$$y_p = Ax^2 + Bx + C$$

$$\therefore \quad y_p' = 2Ax + B$$

$$y_p'' = 2A$$

Substituting in the ODE:

$$2A - 4Ax - 2B + 5Ax^2 + 5Bx + 5C = 10x^2 - 2$$

Equating coefficients of powers of x:

x^2: $\quad 5A = 10 \quad \Rightarrow \quad A = 2$

x: $\quad -4A + 5B = 0 \Rightarrow B = \frac{8}{5}$

x^0: $\quad 2A - 2B + 5C = -2 \Rightarrow C = \frac{1}{5}\left(-2 - 4 + \frac{16}{5}\right) = -\frac{14}{25}$

$$\therefore \quad y_p = 2x^2 + \frac{8}{5}x - \frac{14}{25}$$ (2)

and the complete general solution is

$$y = y_h + y_p = e^x(C_1 \cos 2x + C_2 \sin 2x) + 2x^2 + \frac{8}{5}x - \frac{14}{25}.$$ (1)

5. The auxiliary equation is

$$\lambda^2 - 4\lambda + 3 = 0$$
$$(\lambda - 1)(\lambda - 3) = 0$$
$$\lambda = 1, 3$$

$\therefore$ The solution of the homogeneous equation is

$$y_h = C_1 e^{x} + C_2 e^{3x}.$$ 1

Note that the right hand side of the ODE, e^{3x}, is part of y_h. Therefore we must modify our form for y_p from Ce^{3x} to

$$y_p = C x e^{3x}$$
$$\therefore \quad y_p' = C(1 + 3x)e^{3x}$$
$$y_p'' = C(6 + 9x)e^{3x}$$

Substituting: $C(6 + \cancel{9x} - 4 - \cancel{12x} + \cancel{3x})\, e^{3x} = 4e^{3x}$

(Note that the terms involving x cancel – this must happen.)

$$\therefore \quad 2Ce^{3x} = 4e^{3x} \Rightarrow C = 2$$ 2

$\therefore$ The general solution is

$$y = y_h + y_p = C_1 e^{x} + C_2 e^{3x} + 2x e^{3x}$$

$y(0) = 1 \Rightarrow \quad 1 = C_1 + C_2$ —(i)

$y' = C_1 e^{x} + 3C_2 e^{3x} + 2e^{3x} + 6x e^{3x}$

$y'(0) = 0 \Rightarrow \quad 0 = C_1 + 3C_2 + 2$ —(ii)

Solving (i) + (ii) gives $C_1 = 5/2$, $C_2 = -3/2$.

$$\therefore \quad y = \tfrac{5}{2}e^{x} - \tfrac{3}{2}e^{3x} + 2x e^{3x}.$$ 2