

SS2065: INTRO TO PDES

Summer School, 2012

Assignment 3: to be handed to your tutor during the tutorial on Friday 27th January

1. The equations

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, & 0 < x < \pi, t > 0, \\ u(0, t) &= 0, \\ \frac{\partial u}{\partial x}(\pi, t) &= 0,\end{aligned}$$

model the temperature variation, $u(x, t)$, in a rod of length π , positioned with one end in ice and the other end insulated.

- Find the steady-state solution, $\bar{u}(x)$, to the problem.
- Use the method of separation of variables (set $u(x, y) = \phi(x)G(t)$) and find ODEs for $\phi(x)$ and $G(t)$. (These will involve a parameter λ .)
- Write down the boundary conditions on $\phi(x)$ and solve the ODE for λ and $\phi(x)$.
- Solve the ODE for $G(t)$ and hence write down a solution $u(x, t)$ of the heat equation.
- Show that the general solution satisfying the boundary conditions is

$$u(x, t) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{2n+1}{2}x\right) \exp\left[-\left(\frac{2n+1}{2}\right)^2 t\right].$$

- If we now impose the initial condition $u(x, 0) = f(x)$, $0 \leq x \leq \pi$, show that

$$A_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin\left(\frac{2n+1}{2}x\right) dx.$$

You may assume the result

$$\int_0^{\pi} \sin\left(\frac{2n+1}{2}x\right) \sin\left(\frac{2m+1}{2}x\right) dx = \begin{cases} 0, & n \neq m, \\ \pi/2, & n = m, \end{cases}$$

for n, m integers.

- Find $u(x, t)$ for the case where $f(x) = \pi - x$, $0 \leq x \leq \pi$.