SS 2065 ASSIGNMENT 3 SOLUTIONS

(a) $\frac{\partial \bar{u}(x)}{\partial t} = 0$, so $\frac{\partial^2 \bar{u}(x)}{\partial x^2} = 0$

This is just the ODE $\frac{d^2\bar{u}}{dx^2} = 0$ with solution

$$\bar{u}(x) = Ax + B$$

$u(0,t) = 0 \Rightarrow \bar{u}(0) = 0 \Rightarrow 0 = A \cdot 0 + B = B$

$\therefore \bar{u}(x) = Ax$ and $\frac{d\bar{u}(x)}{dx} = A$.

$\frac{\partial u}{\partial x}(\pi, t) \Rightarrow \frac{d\bar{u}}{dx}(\pi) = 0 \Rightarrow 0 = A$.

$\therefore \bar{u}(x) = 0$.

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[This is obvious on physical grounds - all the heat drains from the $x=0$ end of the rod which is in ice.]

(b) $\phi G' = \phi'' G$

$\therefore \frac{G'}{G} = \frac{\phi''}{\phi} = -\lambda$

(We choose $-\lambda$ for convenience, but $+\lambda$ will also work.)

$\therefore \phi'' + \lambda\phi = 0 \qquad G' = -\lambda G$

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(c) $u(0,t) = 0 \Rightarrow \phi(0) = 0$

$\frac{\partial u}{\partial x}(\pi, t) = 0 \Rightarrow \phi'(\pi) = 0$

Since we have zero boundary conditions, it is clear that an oscillatory (sinusoidal) solution is required.

$\therefore$ Choose $\lambda > 0$, and $\phi'' + \lambda\phi = 0$ has solution

$$\phi(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

$\phi(0) = 0 \Rightarrow \quad 0 = A$

$\therefore \phi(x) = B \sin \sqrt{\lambda} x$

$\phi'(x) = B\sqrt{\lambda} \cos \sqrt{\lambda} x$.

$\phi'(\pi) = 0 \Rightarrow \quad 0 = B\sqrt{\lambda} \cos \sqrt{\lambda}\pi$.

$\lambda \neq 0$, and choosing $B = 0$ gives $\phi(x) = 0$.

$\therefore$ we must choose $\cos\sqrt{\lambda}\pi = 0$

$$\Rightarrow \sqrt{\lambda}\pi = (2n+1)\pi/2, \quad n = 0, 1, 2, \ldots$$

$$\therefore \lambda = \left(\frac{2n+1}{2}\right)^2$$

and $\phi(x) = B \sin\left(\frac{2n+1}{2} x\right)$.

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(d) The ODE for G is now

$$G' = -\left(\frac{2n+1}{2}\right)^2 G$$

Recalling that the equation $\frac{dy}{dt} = \mu y$ has solution $y = Ce^{\mu t}$,

$$G(t) = C e^{-\left(\frac{2n+1}{2}\right)^2 t}$$

$\therefore$ A solution of the heat equation is

$$u(x,t) = \text{Constant} \times \sin\left(\frac{2n+1}{2} x\right) e^{-\left(\frac{2n+1}{2}\right)^2 t}.$$

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(e) From part (d) a solution satisfying the PDE plus BCs is

$$u(x,t) = \text{Constant} \times \sin\left(\frac{2n+1}{2}x\right) e^{-\left(\frac{2n+1}{2}\right)^2 t}, \quad n = 0, 1, 2, \ldots$$

The general solution is a superposition of all these solutions:

$$u(x,t) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{2n+1}{2}x\right) e^{-\left(\frac{2n+1}{2}\right)^2 t} \qquad 1$$

(f) Setting $t=0$:

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{2n+1}{2}x\right)$$

Multiply each side by $\sin\left(\frac{2m+1}{2}x\right)$ and integrate $\int_0^{\pi} \ldots dx$

$$\int_0^{\pi} f(x) \sin\left(\frac{2m+1}{2}x\right) dx = \sum_{n=0}^{\infty} A_n \int_0^{\pi} \sin\left(\frac{2n+1}{2}x\right) \sin\left(\frac{2m+1}{2}x\right) dx$$

$$= \sum_{n=0}^{\infty} A_n \times \begin{cases} 0, & n \neq m \\ \frac{\pi}{2}, & n = m \end{cases}$$

$$= \frac{\pi}{2} A_m$$

$$\therefore A_m = \frac{2}{\pi} \int_0^{\pi} f(x) \sin\left(\frac{2m+1}{2}x\right) dx, \quad m = 0, 1, 2, \ldots$$

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(g) $$A_n = \frac{2}{\pi}\int_0^\pi (\pi - x)\sin\left(\frac{2n+1}{2}x\right)dx \quad , \quad n = 0, 1, 2\ldots$$

$$= \frac{2}{\pi}\left\{\left[(\pi - x)\frac{\cos\left(\frac{2n+1}{2}x\right)}{-\frac{2n+1}{2}}\right]_0^\pi - \int_0^\pi \frac{\cos\left(\frac{2n+1}{2}x\right)}{-\frac{2n+1}{2}}(-1)\,dx\right\}$$

$$= \frac{2}{\pi}\left\{\left[0 - \pi\frac{\cos 0}{-\frac{2n+1}{2}}\right] - \left[\frac{\sin\left(\frac{2n+1}{2}x\right)}{\left(\frac{2n+1}{2}\right)^2}\right]_0^\pi\right\}$$

$$= \frac{2}{\pi}\left\{\frac{2\pi}{2n+1} - \left[\frac{\sin\frac{2n+1}{2}\pi}{\left(\frac{2n+1}{2}\right)^2} - 0\right]\right\}$$

$$= \frac{2}{\pi}\left\{\frac{2\pi}{2n+1} - \left(\frac{2}{2n+1}\right)^2 \sin\left[(2n+1)\frac{\pi}{2}\right]\right\}$$ 3

$$= \frac{2}{\pi}\left\{\frac{2\pi}{2n+1} - \left(\frac{2}{2n+1}\right)^2 \times \begin{cases} 1, & n = 0, 2, 4\ldots \\ -1, & n = 1, 3, 5, \ldots \end{cases}\right]$$

$$\therefore u(x,t) = 4\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\left(\frac{2n+1}{2}x\right)e^{-\left(\frac{2n+1}{2}\right)^2 t}$$

$$- \frac{8}{\pi}\sum_{n=0,2,4,\ldots}^{\infty}\frac{1}{(2n+1)^2}\sin\left(\frac{2n+1}{2}x\right)e^{-\left(\frac{2n+1}{2}\right)^2 t}$$

$$+ \frac{8}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{(2n+1)^2}\sin\left(\frac{2n+1}{2}x\right)e^{-\left(\frac{2n+1}{2}\right)^2 t}$$ 3

[Note: the last two terms could be combined as

$$- \frac{8}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)^2}\sin\left(\frac{2n+1}{2}x\right)e^{-\left(\frac{2n+1}{2}\right)^2 t} \quad]$$