Fourier Transform

This is an integral transform that has some relation to the Laplace transform and also to Fourier series.

Definition.

The Fourier Transform (FT) of a function $f(x)$ is defined to be

$$\boxed{F(\omega) = \mathcal{F}\{f(x)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx}$$

Notes:
- The integral must exist. A necessary condition is $f(x) \to 0$ as $x \to \pm\infty$. A sufficient condition is that $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$
- ω is a real variable. $F(\omega)$ may be complex, even if $f(x)$ is real
- The $\frac{1}{2\pi}$ factor may appear elsewhere in other conventions.

Ex 1. Find the FT of $f(x) = e^{-|x|}$

$$\mathcal{F}\{e^{-|x|}\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|x|}\, e^{i\omega x}\, dx$$

$$=$$

$$= \frac{1}{\pi} \frac{1}{\omega^2+1}$$

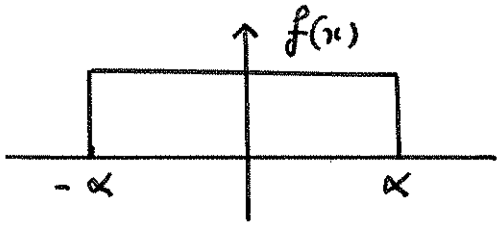
An easy extension of this result is

$$\mathcal{F}\{e^{-\alpha|x|}\} = \frac{1}{\pi} \frac{\alpha}{\omega^2+\alpha^2}, \qquad \alpha > 0$$

[Question: what is $\mathcal{F}\{e^x\}$?]

Ex. 2 Find $\mathcal{F}\{f(x)\}$ if $f(x) = \begin{cases} 1, & -\alpha < x < \alpha \\ 0, & \text{otherwise} \end{cases}, \quad \alpha > 0$

The graph of $f(x)$ is



$$\mathcal{F}\{f(x)\} =$$

$$= \frac{1}{\pi\omega} \sin(\alpha\omega)$$

Ex 3 Show that $\mathcal{F}\{\delta(x-a)\} = \frac{1}{2\pi} e^{i\omega a}, \quad a > 0$.

Fourier transform of derivatives.

$$\mathcal{F}\{f'(x)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f'(x)\, e^{i\omega x}\, dx$$

$$= \frac{1}{2\pi}\left\{ \left[e^{i\omega x} f(x)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\, i\omega\, e^{i\omega x}\, dx \right\}$$

(integration by parts)

$$= \frac{1}{2\pi}\left\{ [0 - 0] - i\omega \int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx \right\}$$

since $f(x) \to 0$ as $x \to \pm\infty$

$$= -i\omega\, F(\omega)$$

Repetition of this type of calculation gives the general result

$$\mathcal{F}\{f^{(n)}(x)\} = (-i\omega)^n F(\omega)$$

where $f^{(n)}(x) \equiv \dfrac{d^n f}{dx^n}$.

[Compare the case of the Laplace transform, where we had results of the type

$$\mathcal{L}\{f'(t)\} = s\,F(s) - f(0)$$

$$\mathcal{L}\{f''(t)\} = s^2 F(s) - s f(0) - f'(0), \quad \text{etc.}$$

Note that the FT case is simpler, as it doesn't involve $f(0)$, $f'(0)$, etc.]

A table of Fourier transforms, together with some basic properties, is given on the next page.

MATH2065: INTRO TO PDEs

Fourier Transforms Table

Function	**Fourier Transform**
$f(x) = \mathcal{F}^{-1}\{F(\omega)\} = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x}\, d\omega$	$F(\omega) = \mathcal{F}\{f(x)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx$
$e^{-\alpha x^2} \quad (\alpha > 0)$	$\frac{1}{\sqrt{4\pi\alpha}} e^{-\omega^2/4\alpha}$
$\sqrt{\frac{\pi}{\beta}} e^{-x^2/4\beta} \quad (\beta > 0)$	$e^{-\beta\omega^2}$
$\frac{2\alpha}{x^2+\alpha^2} \quad (\alpha > 0)$	$e^{-\alpha\|\omega\|}$
$e^{-\alpha\|x\|} \quad (\alpha > 0)$	$\frac{1}{\pi}\frac{\alpha}{\omega^2+\alpha^2}$
$f(x) = \begin{cases} 0 & \|x\| > \alpha \\ 1 & \|x\| < \alpha \end{cases}$	$\frac{1}{\pi}\frac{\sin \alpha\omega}{\omega}$
$\delta(x - x_0)$ (Dirac delta)	$\frac{1}{2\pi} e^{i\omega x_0}$
$a\, f(x) + b\, g(x)$	$a\, F(\omega) + b\, G(\omega)$ (linearity)
$f(x - \beta)$	$e^{i\omega\beta} F(\omega)$ (x-shifting)
$f(x)\, e^{-i\beta x}$	$F(\omega - \beta)$ (ω – shifting)
$x^n f(x)$	$(-i)^n \frac{d^n}{d\omega^n} F(\omega)$ (ω – derivatives)
$\frac{d^n f}{dx^n}$	$(-i\omega)^n F(\omega)$ (x-derivatives)
$f \star g = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x - \bar{x})\, g(\bar{x})\, d\bar{x}$	$F(\omega)\, G(\omega)$ (convolution)

Inverse Fourier Transform.

We write $$f(x) = \mathcal{F}^{-1}\{F(\omega)\}$$

E.g., we found $$\mathcal{F}\{e^{-|x|}\} = \frac{1}{\pi}\frac{1}{\omega^2+1} = F(\omega)$$

so $$\mathcal{F}^{-1}\left\{\frac{1}{\omega^2+1}\right\} = \pi e^{-|x|}.$$

Other inverse transforms can be found from the table on p. 122.
There is also the (somewhat surprising) formula

$$f(x) = \int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega x}\, d\omega.$$

Thus we have the Fourier transform pair:

$$\boxed{\begin{aligned} F(\omega) &= \mathcal{F}\{f(x)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx \\ f(x) &= \mathcal{F}^{-1}\{F(\omega)\} = \int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega x}\, d\omega \end{aligned}}$$

[Relation between Fourier transform and Fourier series.

Recall the complex FS (p. 116)

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L}, \quad f(x+2L) = f(x)$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(\bar{x})\, e^{in\pi\bar{x}/L}\, d\bar{x}$$

$$\therefore\ f(x) = \sum_{n=-\infty}^{\infty}\left(\frac{1}{2L}\int_{-L}^{L} f(\bar{x})\, e^{in\pi\bar{x}/L}\, d\bar{x}\right) e^{-in\pi x/L}$$

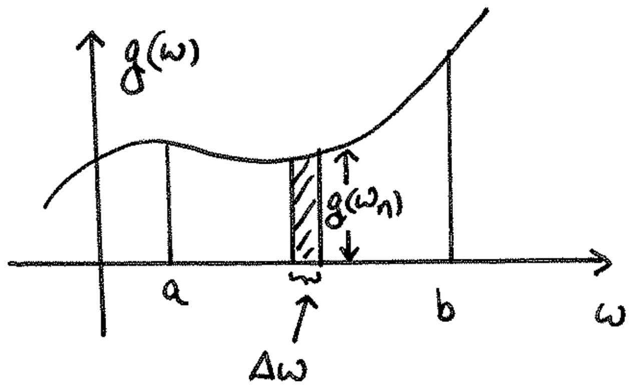
Set $\omega_n = \frac{n\pi}{L}$. Then $\Delta\omega = \omega_{n+1} - \omega_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$

$\therefore L = \frac{\pi}{\Delta\omega}$ and we have

$$f(x) = \sum_{n=-\infty}^{\infty} \left(\frac{\Delta\omega}{2\pi} \int_{-\pi/\Delta\omega}^{\pi/\Delta\omega} f(\bar{x})\, e^{i\omega_n \bar{x}}\, d\bar{x} \right) e^{-i\omega_n x}$$

We now go to the continuous form by letting $L \to \infty$,

$\Rightarrow \Delta\omega \to 0$.



Recall the definition of an integral as a Riemann sum:

$$\int_a^b g(\omega)\, d\omega = \lim_{\Delta\omega \to 0} \sum_{n=1}^{\infty} \Delta\omega\, g(\omega_n)$$

$$= \lim_{\Delta\omega \to 0} \Delta\omega \sum_{n=1}^{\infty} g(\omega_n)$$

$$\therefore f(x) = \lim_{\Delta\omega \to 0} \Delta\omega \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\pi/\Delta\omega}^{\pi/\Delta\omega} f(\bar{x})\, e^{i\omega_n \bar{x}}\, d\bar{x} \right) e^{-i\omega_n x}$$

$$= \int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega x}\, d\omega$$

where $F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\bar{x})\, e^{i\omega \bar{x}}\, d\bar{x}$.

Thus as well as showing that the FT is the continuous analogue of the FS, this also derives the formulas for the Fourier transform pair.]

Some properties of the Fourier transform.

(Refer to the table on p. 122)

1. $\mathcal{F}\{a f(x)\} = a F(\omega)$, a = constant

$\mathcal{F}\{a f(x) + b g(x)\} = a F(\omega) + b G(\omega)$, a, b constants.

This second property is the linearity of the FT.

Both properties follow immediately from the definition of the FT as an integral.

2. $$\mathcal{F}\{f(x-\beta)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} f(x-\beta)\, dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(\bar{x}+\beta)} f(\bar{x})\, d\bar{x}, \quad \bar{x} = x-\beta.$$

$$= e^{i\omega\beta} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\bar{x}} f(\bar{x})\, d\bar{x}$$

$$= e^{i\omega\beta} F(\omega) \qquad (x\text{-shifting})$$

3. $$\mathcal{F}\{f(x) e^{-i\beta x}\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} f(x) e^{-i\beta x}\, dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega-\beta)x} f(x)\, dx$$

$$= F(\omega-\beta). \qquad (\omega\text{-shifting}).$$

4. $$\mathcal{F}\{x f(x)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} x f(x)\, dx$$

Now $$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} f(x)\, dx$$

$$\therefore \quad \frac{d}{d\omega} F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} i x \, e^{i\omega x} f(x) \, dx$$

Multiplying both sides by $-i$ gives

$$-i \frac{d}{d\omega} F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x \, e^{i\omega x} f(x) \, dx \quad (\text{using } -i \times i = +1)$$

$$= \mathcal{F}\{x f(x)\}$$

Repeating this process leads to the general formula

$$\mathcal{F}\{x^n f(x)\} = (-i)^n \frac{d^n}{d\omega^n} F(\omega) .$$

5. $$\mathcal{F}\left\{\frac{df}{dx}\right\} = (-i\omega) F(\omega)$$

(see p. 121). The generalization is

$$\mathcal{F}\left\{\frac{d^n f}{dx^n}\right\} = (-i\omega)^n F(\omega)$$

6. <u>Convolution</u>.

$$\mathcal{F}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x-\bar{x})\, g(\bar{x})\, d\bar{x}\right\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x-\bar{x})\, g(\bar{x})\, d\bar{x}\right) dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\bar{x})\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x-\bar{x})\, e^{i\omega x}\, dx\right) d\bar{x}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\bar{x})\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\, e^{i\omega(y+\bar{x})}\, dy\right) d\bar{x} \quad , \quad y = x-\bar{x}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\bar{x})\, e^{i\omega\bar{x}}\, d\bar{x} \;\; \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\, e^{i\omega y}\, dy = G(\omega) F(\omega)$$

i.e. $$\mathcal{F}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x-\bar{x})\, g(\bar{x})\, d\bar{x}\right\} = F(\omega)\, G(\omega) .$$

Some examples of FTs.

Ex. 1. Find $\mathcal{F}\{ x e^{-|x|} \}$

Answer $= \dfrac{2i\omega}{\pi(\omega^2+1)^2}$

Ex 2 Find $\mathcal{F}\left\{ e^{-3(x-2)^2} \right\}$

Answer $= \dfrac{1}{\sqrt{12\pi}} e^{-\omega^2/12} e^{2i\omega}$

Ex 3 Find $\mathcal{F}\{ e^{-i\beta x} \}$

Answer $= \delta(\beta - \omega)$.

Use of FTs in solving PDEs.

FTs can be useful in solving PDEs where one of the variables has an infinite range. We illustrate the method with a number of examples.

Ex.1. Find $u = u(x,t)$ if

$$\frac{\partial u}{\partial t} = 3\frac{\partial u}{\partial x}, \quad -\infty < x < \infty, \quad t > 0$$

subject to $u(x,0) = f(x)$, where $f(x)$ is some given function.

We take a FT with respect to x. Let

$$U(\omega, t) = \mathcal{F}\{u(x,t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)e^{i\omega x}dx$$

Then the FT of the PDE is

$$\frac{dU(\omega,t)}{dt} = 3(-i\omega)U(\omega,t)$$

where we have use $\mathcal{F}\left\{\frac{\partial u}{\partial x}\right\} = -i\omega U(\omega,t)$.

[Note that we write $\mathcal{F}\left\{\frac{\partial u}{\partial t}\right\} = \frac{dU(\omega,t)}{dt}$, rather than $\frac{\partial U(\omega,t)}{\partial t}$. This is because $u = u(x,t)$ is a function of the two variables x and t, and x could depend on t so we must use $\frac{\partial}{\partial t}$ to indicate that we keep x constant when differentiating. $U = U(\omega,t)$ is a function of ω and t, but ω is a parameter that cannot depend on t, so we can use the ordinary derivative, $\frac{d}{dt}$]

Thus we now have to solve the ODE

$$\frac{dU}{dt} = -3i\omega U$$

Recall $\frac{dy}{dt} = \alpha y \Rightarrow y = Ae^{\alpha t}$ where A is a constant independent of t. Thus

$$U(\omega,t) = A(\omega)e^{-3i\omega t}$$

where $A(\omega)$ is independent of t, but can depend on ω.

When $t=0$, $\quad U(\omega,0) = A(\omega)$

But $\quad u(x,0) = f(x)$ and taking the FT gives

$$U(\omega,0) = \mathcal{F}\{f(x)\} = F(\omega)$$

$$\therefore \quad A(\omega) = F(\omega)$$

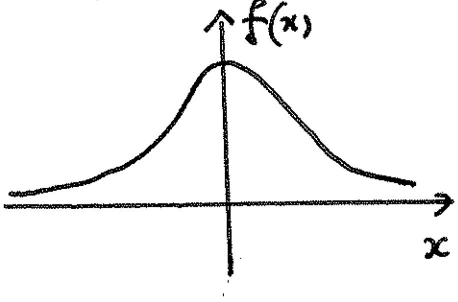
and $\quad U(\omega,t) = F(\omega)e^{-3i\omega t}$

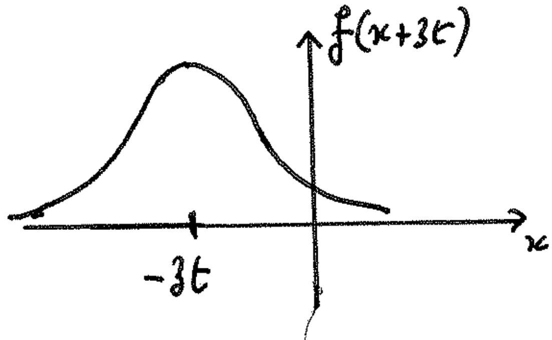
From the table (p. 122), $\mathcal{F}^{-1}\{e^{i\omega\beta}F(\omega)\} = f(x-\beta)$

$$\therefore \quad u(x,t) = \mathcal{F}^{-1}\{e^{-3i\omega t}F(\omega)\}$$
$$= f(x+3t).$$

This can be interpreted as some initial configuration $u(x,0) = f(x)$ that travels to the left with the same shape and speed 3.

e.g.





Ex 2 Find $u = u(x,t)$ satisfying

$$\frac{\partial^2 u}{\partial t\,\partial x} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \ t > 0$$

with initial condition $u(x,0) = \pi e^{-|x|}$.

We take the FT of this PDE with respect to x:

$$\frac{d}{dt}\left[-i\omega\, U(\omega,t)\right] = (-i\omega)^2\, U(\omega,t)$$

Solving and using the initial condition:

$$U(\omega,t) = \frac{1}{\omega^2+1}\, e^{-i\omega t}$$

Inverting the FT:

$$u(x,t) = \pi e^{-|x+t|}$$

[Note: The FT makes the implicit assumption that $F(\omega)$ exists, so it assumes $f(x) \to 0$ as $x \to \pm\infty$. Any solution not satisfying this requirement will not be found by the FT method. Thus it acts as a filter and removes unphysical solutions.]

Fourier transform of a Gaussian function.

The function

$$f(x) = e^{-\alpha x^2}, \quad \alpha > 0$$

is called a Gaussian, and appears in numerous applications.

Its FT is

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} e^{-\alpha x^2}\,dx$$

This integral can be done by contour integral methods, but here we adopt an indirect approach. The derivative of $f(x)$ is

$$f'(x) = -2\alpha x e^{-\alpha x^2} = -2\alpha x f(x)$$

The FT of this is

$$-i\omega F(\omega) = -2\alpha(-i)\frac{dF(\omega)}{d\omega}$$

or

$$\frac{dF}{F} = -\frac{\omega}{2\alpha}\,d\omega$$

Integrating,

$$\ln|F| = -\frac{1}{2\alpha}\frac{\omega^2}{2} + \ln|C|, \quad C = \text{constant}$$

$$\therefore \quad |F| = |C|e^{-\omega^2/4\alpha}$$

or

$$F(\omega) = K e^{-\omega^2/4\alpha} \quad \text{where } K = \pm|C|.$$

Now

$$F(0) = K$$

But

$$F(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 1\, e^{-\alpha x^2}\,dx = \frac{1}{2\pi\sqrt{\alpha}}\int_{-\infty}^{\infty} e^{-z^2}\,dz$$

$$(z = \sqrt{\alpha}\,x)$$

and using the standard result $\int_{-\infty}^{\infty} e^{-z^2}\,dz = \sqrt{\pi}$

we get $K = F(0) = \frac{1}{2\pi\sqrt{\alpha}}\sqrt{\pi} = \frac{1}{\sqrt{4\pi\alpha}}$

Hence

$$\mathcal{F}\{e^{-\alpha x^2}\} = F(\omega) = \frac{1}{\sqrt{4\pi\alpha}}\, e^{-\omega^2/4\alpha}.$$

Ex 3. Heat equation on an infinite interval.

Solve $$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t>0$$

subject to the initial condition $u(x,0) = f(x)$.

The FT of the PDE is

$$\frac{dU}{dt} = k(-i\omega)^2 U, \quad U \equiv U(\omega,t) = \mathcal{F}\{u(x,t)\}$$

Solving and applying the initial condition:

$$U(\omega,t) = F(\omega)e^{-k\omega^2 t}$$

$$\therefore u(x,t) = \mathcal{F}^{-1}\{U(\omega,t)\} = \int_{-\infty}^{\infty} F(\omega)e^{-k\omega^2 t}e^{-i\omega x}\,d\omega$$

For anything but the simplest $F(\omega)$, this integral would have to be done numerically and this can be problematic, since $e^{-i\omega x} = \cos\omega x - i\sin\omega x$ is an oscillatory function and thus we get cancellation between positive and negative contributions, so high numerical accuracy is needed. We can get a much nicer formula by using the convolution formula.

Let $G(\omega) = e^{-k\omega^2 t}$

Then $$u(x,t) = \mathcal{F}^{-1}\{F(\omega)G(\omega)\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x-\bar{x})\,g(\bar{x})\,d\bar{x}$$

By symmetry, this is also

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})\, g(x-\bar{x})\, d\bar{x}$$

Now $\quad g(x) = \mathcal{F}^{-1}\{G(\omega)\} = \mathcal{F}^{-1}\{e^{-k\omega^2 t}\}$

$$= \sqrt{\frac{\pi}{kt}}\, e^{-x^2/4kt} \qquad \text{(see Table, p. 122)}$$

$$\therefore \quad u(x,t) = \frac{1}{2\pi}\sqrt{\frac{\pi}{kt}}\int_{-\infty}^{\infty} f(\bar{x})\, e^{-(x-\bar{x})^2/4kt}\, d\bar{x}$$

Note that this form of the solution has two advantages:

1) It involves the original function $f(x)$, so there is no need to calculate $F(\omega)$

2) The integrand does not oscillate, but decreases rapidly as $\bar{x} \to \pm\infty$, so it is easy to do numerically.

<u>Ex. 4.</u> Solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad -\infty < x < \infty, \; y > 0$$

for $u = u(x,y)$, subject to $u(x,0) = f(x)$.

(u could be the steady-state temperature in a half-infinite conducting plate.)

Let $U(\omega, y) = \mathcal{F}\{u(x,y)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\, u(x,y)\, dx$

Taking the FT of the PDE gives the ODE:

which has solution

$$U(\omega, y) =$$

We now apply the condition that $U(\omega, y)$ must remain finite for all y, and in particular as $y \to \infty$:

$$\therefore U(\omega, y) =$$

Applying the initial condition $u(x, 0) = f(x)$:

$$U(\omega, y) = F(\omega)\, e^{-|\omega| y}, \quad y > 0 .$$

Inverting the transform using the convolution formula

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} f(\bar{x}) \frac{1}{(x-\bar{x})^2+y^2}\, d\bar{x}$$

Specific choices for $f(x)$:

(a) $f(x) = \delta(x)$

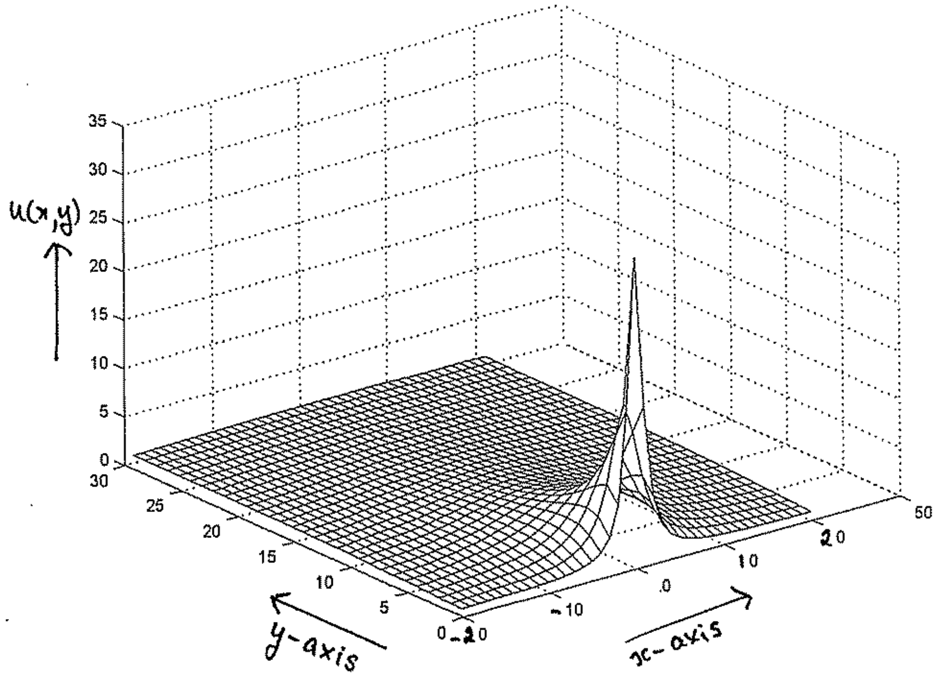
$$\Rightarrow u(x,y) = \frac{1}{\pi}\frac{y}{x^2+y^2}$$

(b) $f(x) = \begin{cases} 0, & x<0 \\ 1, & x>0 \end{cases}$

$$\Rightarrow u(x,y) =$$

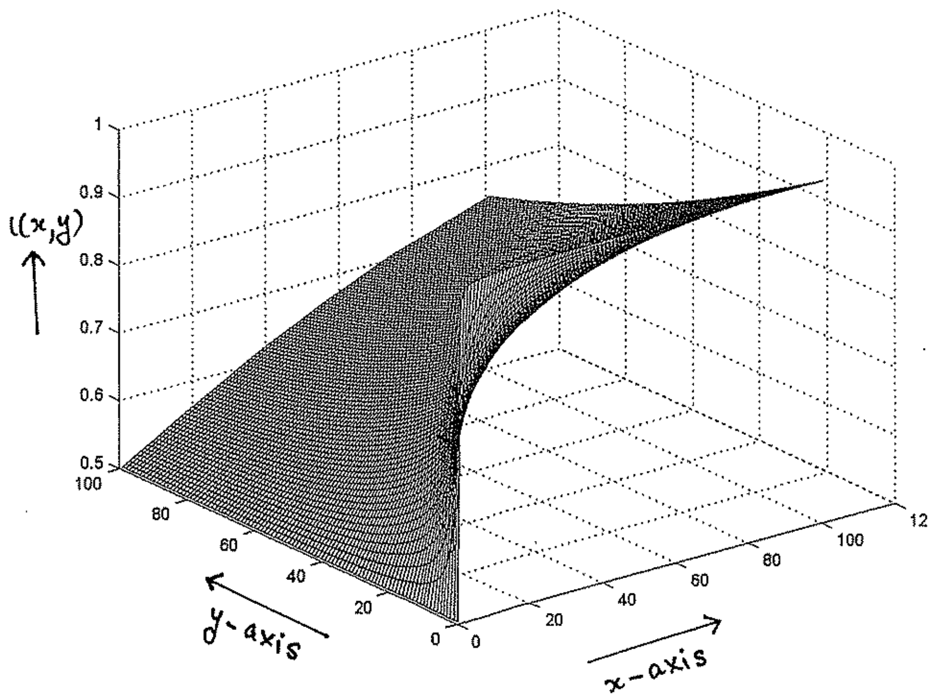
$$= \frac{1}{\pi}\left(\frac{\pi}{2} + \tan^{-1}\left(\frac{x}{y}\right)\right)$$

See graphs on the next page.



$$u(x,y) = \frac{1}{\pi} \frac{y}{x^2+y^2}.$$

Since $u(x,y)$ is infinite at $(x,y)=(0,0)$, the graph should show a spike of infinite height at $(0,0)$, but only part of this spike can be illustrated.



$$u(x,y) = \frac{1}{\pi}\left(\frac{\pi}{2} + \tan^{-1}\left(\frac{x}{y}\right)\right).$$

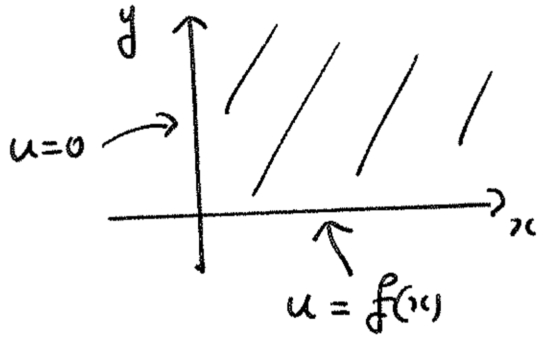
Note that

$$u(0,y) = \frac{1}{\pi}\cdot\frac{\pi}{2} = \frac{1}{2}$$
$$= \frac{1}{2}\left(f(0+) + f(0-)\right)$$

Ex 5.

Solve $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, $\quad 0 < x < \infty$, $\; 0 < y < \infty$.

subject to $u(x,0) = f(x)$, $u(0,y) = 0$.



This could give the steady-state temperature in a quarter-plate with one edge kept at 0 temperature and the other edge at a prescribed temperature $f(x)$.

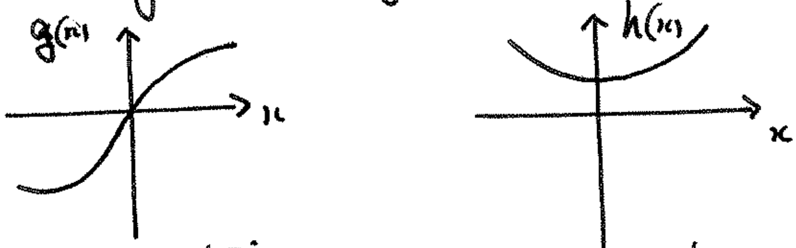
$u(x,y)$ is not defined for $x<0$ (or $y<0$), so in order to use the FT we need to extend it. Because of the boundary condition $u(0,y)=0$ it is sensible to extend $u(x,y)$ as an odd function.

I.e., let $u(-x,y) = -u(x,y)$ for all x, $-\infty < x < \infty$.

[Let $g(x)$ be any odd function of x that is continuous at $x=0$.

Now $g(-x) = -g(x)$

and setting $x=0$ gives $g(0) = -g(0) \Rightarrow g(0) = 0$



odd, continuous $\qquad\qquad$ even, continuous.

But an even function $h(x)$ does not have to be zero at $x=0$.

$\therefore$ Since $u(x,y)=0$ when $x=0$ an odd extension is best.]

The FT of $u(x,y)$ is

$$U(\omega,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,y)e^{i\omega x}\,dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{u(x,y)}_{\text{odd in } x}(\underbrace{\cos\omega x}_{\text{even}} + i\underbrace{\sin\omega x}_{\text{odd}})\,dx$$

$$= \frac{i}{\pi}\int_{0}^{\infty} u(x,y)\sin\omega x\,dx$$

(This is called a Fourier sine transform.)

Now take the FT of the PDE and solve for $U(\omega,y)$:

$U(\omega,y) =$

Applying the condition that $U(\omega,y)$ is finite as $y\to\infty$:

$U(\omega,y) = C(\omega)\,e^{-|\omega|y}$

Now $U(\omega,0) = \mathcal{F}\{f(x)\}$

But $f(x)$ is only defined for $x>0$. To make things consistent, we extend $f(x)$ as an odd function:

$$f_{odd}(x) = \begin{cases} f(x), & x>0 \\ -f(-x), & x<0. \end{cases}$$

Then $U(\omega,0) = \mathcal{F}\{f_{odd}(x)\} = \frac{i}{\pi}\int_0^\infty f(x)\sin\omega x\, dx = F_{odd}(\omega)$.

$$\therefore U(\omega,y) = F_{odd}(\omega)\, e^{-|\omega| y}$$

Inverting the FT, using the convolution formula:

$$u(x,y) = \frac{1}{2\pi}\int_0^\infty f(\bar{x})\left[\frac{2y}{(x-\bar{x})^2+y^2} - \frac{2y}{(x+\bar{x})^2+y^2}\right] d\bar{x}$$