SOLUTIONS.

MATH2065: INTRODUCTION TO PDEs

Summer School 2012

Quiz 1

NAME:

STUDENT NUMBER:

Time allowed: 50 minutes

This quiz is worth 10% of the final mark for the course.
There are 15 questions, each worth the same number of marks.

Space is provided after each question for your working and answer.

A table of Laplace transforms is provided.

1. Find the general solution $y(x)$ of the following differential equation:

$$y'' + 3y' + 2y = 0.$$

AE :

$$\lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda + 1)(\lambda + 2) = 0$$

$$\lambda = -1, -2$$

$$\therefore \; y = C_1 e^{-x} + C_2 e^{-2x}$$

2. Find a particular solution $y_p(x)$ of the following differential equation:

$$y'' - 3y' + 2y = x - 3$$

Try $y_p = ax + b$

$y_p' = a$

$y_p'' = 0$

Substituting :

$0 - 3a + 2ax + 2b = x - 3$

Equating coeffs of powers of x:

x : $2a = 1 \Rightarrow a = 1/2$

x^0: $-3a + 2b = -3$

$\therefore 2b = -3 + 3/2 = -3/2$

$\therefore b = -\frac{3}{4}$

$y_p = \frac{1}{2}x - \frac{3}{4}$

3. Find the general solution $y(x)$ of the following differential equation:

$$y'' + 2y' + 3y = 0.$$

Express your answer in terms of real quantities.

AE: $\lambda^2 + 2\lambda + 3 = 0$

$$\lambda = \frac{-2 \pm \sqrt{4-12}}{2}$$

$$= \frac{-2 \pm \sqrt{-8}}{2}$$

$$= \frac{-2 \pm i2\sqrt{2}}{2}$$

$$= -1 \pm i\sqrt{2}$$

$$\therefore\ y = e^{-x}\left(C_1 \cos(\sqrt{2}x) + C_2 \sin(\sqrt{2}x)\right)$$

4. You are **given** that the differential equation

$$y'' + 4y' + 4y = 9e^x$$

has homogeneous solution $y_h = C_1e^{-2x} + C_2xe^{-2x}$ and particular solution $y_p = e^x$. Find the solution satisfying $y(0) = 0$ and $y'(0) = 0$.

$$y(x) = C_1 e^{-2x} + C_2 x e^{-2x} + e^x$$

$$y(0) = 0 \Rightarrow \quad 0 = C_1 + 1 \quad \Rightarrow \quad C_1 = -1$$

$$y'(x) = -2C_1 e^{-2x} + C_2 e^{-2x} - 2C_2 x e^{-2x} + e^x$$

$$y'(0) = 0 \Rightarrow \quad 0 = -2C_1 + C_2 + 1$$

$$\therefore \quad C_2 = 2C_1 - 1 = -2 - 1 = -3$$

$$y = -e^{-2x} - 3x e^{-2x} + e^x$$

5. Using the Table, or otherwise, find the Laplace transform of t^4e^{2t}.

$$\mathcal{L}\{e^{2t}\,t^4\} = F(s-2)$$

where $F(s) = \mathcal{L}\{t^4\}$

$$= \frac{4!}{s^5}$$

$$\therefore \mathcal{L}\{e^{2t}\,t^4\} = \frac{4!}{(s-2)^5} \quad \left(= \frac{24}{(s-2)^5}\right) \quad (s>2)$$

6. Using the Table, or otherwise, find the Laplace transform of $H(t-5)e^{3t}$, where $H(t)$ is the Heaviside unit-step function.

$$\mathcal{L}\{H(t-5)e^{3t}\} = \mathcal{L}\{H(t-5)e^{3(t-5)}e^{15}\}$$

$$= e^{15}e^{-5s}\mathcal{L}\{e^{3t}\}$$

$$= e^{15}e^{-5s}\frac{1}{s-3} \qquad (s>3)$$

7. Using the Table, or otherwise, find the inverse Laplace transform of $\frac{s}{s^2+2s-3}$.

$$\frac{s}{s^2+2s-3} = \frac{s}{(s-1)(s+3)}$$

$$= \frac{\frac{1}{4}}{s-1} + \frac{\frac{3}{4}}{s+3}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{s}{s^2+2s-3}\right\} = \frac{1}{4}e^{t} + \frac{3}{4}e^{-3t}$$

8. Using Laplace transforms, or otherwise, solve

$$y'' + 2y' + 5y = 0$$

for $y(t)$, given that $y(0) = 2$, $y'(0) = -4$.

Let $Y(s) = \mathcal{L}\{y(t)\}$

Taking the LT of the DE:

$$s^2 Y - s\,y(0) - y'(0) + 2[sY - y(0)] + 5Y = 0$$

$$(s^2 + 2s + 5)Y - 2s + 4 - 4 = 0$$

$$Y = \frac{2s}{s^2 + 2s + 5}$$

$$= \frac{2(s+1) - 2}{(s+1)^2 + 4}$$

$$\therefore \quad y(t) = e^{-t}\,\mathcal{L}^{-1}\left\{\frac{2s - 2}{s^2 + 4}\right\}$$

$$= e^{-t}\left(2\cos(2t) - \sin(2t)\right)$$

9. Use Laplace transforms to solve

$$y'' + 6y' + 9y = g(t)$$

for $y(t)$, given that $y(0) = 0$, $y'(0) = 0$. Use the convolution theorem to express the answer as an integral involving $g(t)$.

LT of DE gives

$$s^2 Y + 6sY + 9Y = G(s)$$

where $G(s) = \mathcal{L}\{g(t)\}$

$$\therefore Y = \frac{1}{(s+3)^2} G(s)$$

$$= F(s)\, G(s)$$

where $F(s) = \dfrac{1}{(s+3)^2}$

$$\therefore y(t) = \int_0^t f(t-\tau) g(\tau)\, d\tau$$

where $f(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{(s+3)^2}\right\}$

$$= e^{-3t} \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= e^{-3t}\, t$$

$$\therefore y(t) = \int_0^t e^{-3(t-\tau)}(t-\tau)\, g(\tau)\, d\tau$$

(Also $y(t) = \int_0^t e^{-3\tau} \tau\, g(t-\tau)\, d\tau$).

10. Find the general solution of the partial differential equation

$$\frac{\partial^2 u}{\partial y \partial x} = 1$$

where $u = u(x, y)$.

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 1$$

$$\therefore \quad \frac{\partial u}{\partial x} = y + f(x)$$

$$u = xy + \int f(x)\,dx + g(y)$$

$$= xy + h(x) + g(y)$$

where $h(x)$ and $g(y)$ are arbitrary functions.

11. Find the ordinary differential equations that result from applying the method of separation of variables to the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^3 u}{\partial t^3} = 0.$$

[Let $u(x,t) = X(x)T(t)$ and find find differential equations for X and T.]

$$X''T + XT''' = 0$$

$\div XT$
$$\frac{X''}{X} + \frac{T'''}{T} = 0$$

$$\therefore \frac{X''}{X} = -\frac{T'''}{T} = \lambda = \text{constant}.$$

the DEs are

$$X'' - \lambda X = 0, \qquad T''' + \lambda T = 0$$

12. The function

$$\phi(x) = A\cos(px) + B\sin(px)$$

where A, B and $p(> 0)$ are constants, is required to satisfy the boundary conditions $\phi'(0) = 0$ and $\phi(h) = 0$, where h is a given positive constant. Find the values of p that give a non-zero solution for ϕ.

$$\phi'(x) = -pA\sin(px) + pB\cos(px)$$

$$\phi'(0)=0 \Rightarrow \quad 0 = 0 + pB$$

$$\therefore B = 0$$

$$\phi(h)=0 \Rightarrow \quad 0 = A\cos(ph)$$

For a non-zero solution,

$$\cos(ph) = 0$$

$$\therefore ph = (n+\tfrac{1}{2})\pi, \quad n = 0, 1, 2, \ldots$$

$$p = (n+\tfrac{1}{2})\frac{\pi}{h}.$$

13. Given that $\phi(x)$ satisfies

$$\frac{d^2\phi}{dx^2} = -\lambda\phi$$

and the boundary conditions $\phi'(0) = 0$, $\phi'(L) = 0$ where $L > 0$, find λ and ϕ.

set $\lambda = p^2$, $p > 0$.

Then $\phi(x) = A\cos(px) + B\sin(px)$

$\phi'(x) = -pA\sin(px) + pB\cos(px)$

$\phi'(0)=0 \Rightarrow \quad 0 = pB \Rightarrow B = 0$

$\phi'(L)=0 \Rightarrow \quad 0 = -pA\sin(pL)$

$\therefore \quad pL = n\pi, \quad n = 1, 2, \ldots$

$$\left.\begin{aligned} \lambda &= \left(\frac{n\pi}{L}\right)^2 \\ \phi(x) &= A\cos(px) = A\cos\left(\frac{n\pi x}{L}\right) \end{aligned}\right\} \quad n = 1, 2, 3, \ldots$$

14. A rod of length L has its end $x = 0$ kept at a temperature of $100°$ and its other end $x = L$ is insulated. If $u(x,t)$ is the temperature at the point x on the rod at time t, write down the boundary conditions that must be satisfied by $u(x,t)$.

$$u(0,t) = 100$$

$$\frac{\partial u}{\partial x}(L,t) = 0$$

15. You are **given** that the solution to the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ for the rod described by $0 < x < 1$ with insulated boundary conditions is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x) e^{-n^2\pi^2 t}$$

for some constants $A_0, A_1, \ldots$. Find the constants for the case where $u(x,0) = 2+6\cos(3\pi x)$.

$A_0 = 2$, $A_3 = 6$, $A_n = 0$ otherwise.