MATH2065: INTRO TO PDEs

Summer School 2012

Tutorial Questions 10

Questions marked with the dagger symbol † are intentionally more challenging.

1. (a) Calculate the Fourier series for the 2π-periodic extension of the function

$$f(x) = x^2, \qquad -\pi < x \le \pi\,.$$

Answer: $\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$.

(b) From part (a), deduce that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}\,.$$

(c) Is it correct to evaluate the Fourier series at $x = 3\pi$, and thereby conclude that

$$(3\pi)^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(3n\pi) \quad ?$$

Why or why not?

(d) Differentiate your result in part (a), and thereby determine the Fourier series for the (2π-periodic extension of the) function x.

(e) Is it possible to differentiate your result in (d), in order to obtain the Fourier series for the (2π-periodic extension of the) function 1?

2. Consider the function

$$f(x) = \begin{cases} 0 & x < x_0 \\ 1/\Delta & x_0 < x < x_0 + \Delta \\ 0 & x > x_0 + \Delta \end{cases}$$

Assume that $x_0 > -L$ and $x_0 + \Delta < L$. Determine the complex Fourier coefficients c_n. *Hint:* The coefficients c_n are given by $c_n = \frac{1}{2}(a_n + ib_n) = \frac{1}{2L}\int_{-L}^{L} f(x) e^{in\pi x/L}\,dx$.

3. † Let f be a $2L$-periodic real function with complex Fourier coefficients c_n. Derive the complex form of Parseval's identity:

$$\frac{1}{2L}\int_{-L}^{L} |f(x)|^2 \,\mathrm{d}x = \sum_{n=-\infty}^{\infty} |c_n|^2\,.$$

Hint: The correct form of the inner-product between *complex-valued* functions ϕ and ψ is given by

$$\langle \phi(x), \psi(x) \rangle := \int_{-L}^{L} \overline{\phi(x)}\,\psi(x)\,\mathrm{d}x\,,$$

where the overbar indicates the complex conjugate.