MATH2065: INTRO TO PDEs

Summer School 2012

Tutorial Solutions 10

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1. (a) The b_n Fourier coefficients are all zero since $f(x)$ is even. The period of the function here is 2π, and hence we use our standard Fourier series formulas with $2L = 2\pi$, i.e., with $L = \pi$. Thus,

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx \\ &= \frac{2}{2\pi}\int_0^{\pi} x^2\,dx \qquad \text{(even integrand)} \\ &= \frac{1}{\pi}\frac{x^3}{3}\bigg|_0^{\pi} \\ &= \frac{\pi^2}{3}\,. \end{aligned}$$

Similarly for $n \neq 0$, the standard Fourier formula gives

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx \qquad \text{(even integrand)} \\ &= \frac{2}{\pi}\int_0^{\pi}\frac{x^2}{n}\,d\,(\sin nx) \\ &= \frac{2}{\pi}\left[\frac{x^2}{n}\sin nx\bigg|_0^{\pi} + \int_0^{\pi}\frac{2x}{n^2}\,d\,(\cos nx)\right] \\ &= \frac{2}{\pi}\left[\frac{2x\,\cos nx}{n^2}\bigg|_0^{\pi} - \int_0^{\pi}\frac{2}{n^2}\cos nx\,dx\right] \\ &= \frac{2}{\pi}\frac{2\pi(-1)^n}{n^2} - \frac{2}{\pi}\frac{2}{n^2}\frac{\sin nx}{n}\bigg|_0^{\pi} \\ &= \frac{4}{n^2}(-1)^n\,. \end{aligned}$$

after integrating twice by parts, and utilising the standard results $\sin nx = 0$ and $\cos nx = (-1)^n$ for integers n. Hence the Fourier series for this function is

$$\frac{\pi^2}{3} - 4\left(\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \cdots\right) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx\,.$$

(b) Since $f(x)$ is a continuous function and $f(0) = 0$, the Fourier series at $x = 0$ converges to 0. See the figure in part (c). Therefore, by substituting $x = 0$ into the result in part (a), we find

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos 0 = 0\,,$$

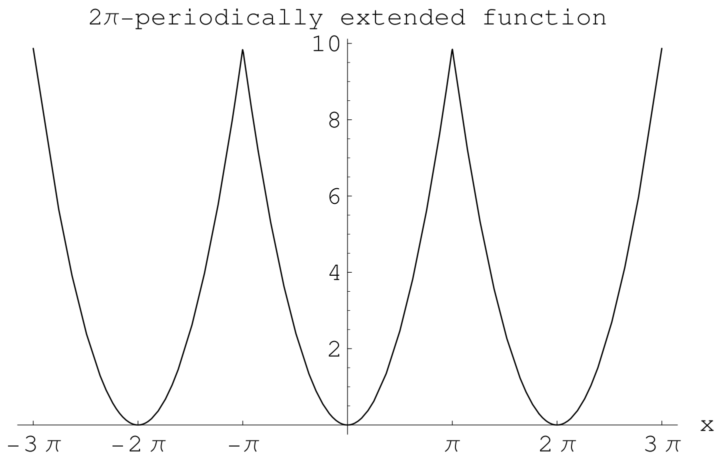
and hence

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}\,.$$

Multiplication by (-1) gives the desired result. This means that

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{1}{12}\pi^2 .$$

(c) The function $f(x)$ is continuous, and therefore it is legitimate to say that the Fourier series *equals* the function. However, inspection of the figure shows that the value of the function at $x = 3\pi$ is *not* $(3\pi)^2$, but is in fact the same as the value of the function at $x = \pi$, i.e., π^2. The expression $f(x) = x^2$ is *only* valid for the function in the domain $(-\pi, \pi)$, and the rest of the function is obtained by extending this segment periodically, with period 2π. Hence, the expression given would be valid if a π^2 appeared on the left hand side.



(d) On the interval $(-\pi, \pi)$, we have that

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx .$$

Differentiating this term-by-term, we get

$$2x = -4\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx \qquad \Rightarrow \qquad x = -2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx ,$$

which gives us the Fourier series for the function x. The equals sign is valid in the domain $(-\pi, \pi)$, in which the function is continuous – but it will be necessary to be careful if going beyond this domain, since the 2π-extention of the function $f(x) = x$ on $(-\pi, \pi)$ is not continuous at $\pm\pi$. So the Fourier series will actually not be *equal* to the 2π-periodic extension of the function at these values, but will take on the average of the left and right limits. This problem did not occur for the 2π-periodic extension of the function x^2, since this is continuous from the figure.

(e) Suppose we differentiated the Fourier series we obtained above in part (d) term by term, to get the expression

$$1 \stackrel{?}{=} -2\sum_{n=1}^{\infty} (-1)^n \cos nx .$$

This is clearly not valid for all $x \in (-\pi, \pi)$, since the function 1 is just the function 1, whose Fourier series is just 1, and not the entity on the right side above. Indeed, this infinite series above does not converge, since its terms do not eventually decay to zero. The point is that it is *not* legitimate to differentiate the Fourier series of the function x in order to attempt to obtain the Fourier series of the function 1, because the 2π-extension of the function x is *not* continuous. For a term-by-term differentiation to be valid, it is necessary that the function be *continuous*, with its derivative being *piecewise-continuous*. This was valid in part (d) (as is clear by inspecting the given graph of the function), but not here.

2. The complex Fourier coefficients as given in the hint are

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\exp\left(\frac{in\pi x}{L}\right)dx = \frac{1}{2L\Delta}\int_{x_0}^{x_0+\Delta}\exp\left(\frac{in\pi x}{L}\right)dx\,.$$

Thus,

$$c_n = \frac{1}{2L\Delta}\frac{L}{in\pi}\exp\left(\frac{in\pi x}{L}\right)\Big|_{x=x_0}^{x_0+\Delta} = \frac{1}{2in\pi\Delta}\exp\left(\frac{in\pi x_0}{L}\right)\left[\exp\left(\frac{in\pi\Delta}{L}\right)-1\right].$$

Equivalently,

$$\begin{aligned} c_n &= \frac{1}{2in\pi\Delta}\exp\left(\frac{in\pi(x_0+\Delta/2)}{L}\right)\left[\exp\left(\frac{in\pi\Delta}{2L}\right)-\exp\left(\frac{-in\pi\Delta}{2L}\right)\right] \\ &= \frac{1}{n\pi\Delta}\exp\left(\frac{in\pi(2x_0+\Delta)}{2L}\right)\sin\left(\frac{n\pi\Delta}{2L}\right). \end{aligned}$$

3. We begin this derivation with two observations concerning complex numbers. First, we note that for any complex number z,

$$z\,\overline{z} = |z|^2\,,$$

that is, the modulus of the complex number can be obtained by multiplying the complex number with its conjugate, and then taking the square-root. We will not prove this simple fact, since it is trivial to do so by writing $z = a + ib$.
Secondly, we note that

$$\overline{e^{-i\theta}} = \overline{\cos\theta - i\sin\theta} = \cos\theta + i\sin\theta = e^{i\theta}\,,$$

for any real θ.
Now, we are ready to derive Parseval's formula. Recall that the complex Fourier representation of the function f is given by

$$f \sim \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L}\,.$$

Consider the function f, and takes its (complex) inner product with the above expression. Then we get

$$\begin{aligned} \int_{-L}^{L}|f(x)|^2\,dx &= \sum_{n=-\infty}^{\infty} c_n\int_{-L}^{L} f(x)e^{-in\pi x/L}\,dx \\ &= 2L\sum_{n=-\infty}^{\infty} c_n\overline{c_n}\,, \end{aligned}$$

where we have used the fact that

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\,e^{in\pi x/L}\,dx\,,$$

and hence $\overline{c_n}$ is the same integral but with the *negative* exponential. Dividing our result by $2L$ gives us the complex form of Parseval's identity. Indeed, Parseval's identity as stated is even valid for *complex-valued* functions f.