MATH2065: INTRO TO PDEs

Summer School 2012

Tutorial Questions 13

This Tutorial Set contains a selection of questions covering material over the semester.

1. Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad ; \quad -\infty < x < \infty \,,$$
$$u(x,0) = f(x)\,,$$
$$\frac{\partial u}{\partial t}(x,0) = 0\,.$$

   (a) Let $U(\omega, t) = \mathcal{F}\{u(x,t)\}$, that is, the Fourier transform of the solution $u(x,t)$ with respect to x, while treating t as a parameter. By Fourier transforming the given PDE, show that

$$\frac{\partial^2 U}{\partial t^2} + c^2\omega^2 U(\omega,t) = 0\,.$$

   (b) Obtain the fact that the general solution to the differential equation in part (a) is

$$U(\omega,t) = A(\omega)\cos(c\omega t) + B(\omega)\sin(c\omega t)\;,$$

   for some arbitrary functions $A(\omega)$ and $B(\omega)$.

   (c) Let $F(\omega) = \mathcal{F}\{f(x)\}$. Show that $B(\omega) = 0$ and $A(\omega) = F(\omega)$.

   (d) Thereby, obtain that the solution to the PDE can be written as

$$u(x,t) = \int_{-\infty}^{\infty} F(\omega)\cos(c\omega t)\, e^{-i\omega x}\, d\omega\,.$$

   (e) Using the fact that $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$, show that the solution is also representible as

$$u(x,t) = \frac{1}{2}\int_{-\infty}^{\infty} F(\omega) e^{-i\omega(x-ct)}\, d\omega + \frac{1}{2}\int_{-\infty}^{\infty} F(\omega) e^{-i\omega(x+ct)}\, d\omega\,,$$

   and hence deduce that the solution can be written in the very simple form

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right]\,.$$

2. This question relates to solving the modified wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{1}{L^2}u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{1}$$

   for the unknown $u(x,t)$ within the domain $0 < x < L$ for positive times $t > 0$. Here, L and c are known positive constants. This equation is subject to the boundary and initial conditions

$$\begin{aligned} u(0,t) &= 0 \quad \text{for all} \;\; t \ge 0\,, \\ u(L,t) &= 0 \quad \text{for all} \;\; t \ge 0\,, \\ u(x,0) &= 0 \quad \text{for all} \;\; x \in [0,L]\,, \\ \frac{\partial u}{\partial t}(x,0) &= f(x) \quad \text{for all} \;\; x \in [0,L]\,, \end{aligned}$$

   where $f(x)$ is a given function. We will go through a step-by-step solution, using our standard methods.

(a) Postulating a separable solution of the form $u(x,t) = X(x)\,T(t)$, and considering all homogeneous (i.e., zero) conditions, obtain the equations

$$X''(x) - \left[\lambda - \frac{1}{L^2}\right] X(x) = 0 \; ; \; X(0) = X(L) = 0 \,, \tag{2}$$

$$T''(t) - \lambda\, c^2\, T(t) = 0 \; ; \; T(0) = 0, \tag{3}$$

where λ is a separation constant.

(b) Considering the equation (2) for $X(x)$, show that its eigenvalues are

$$\lambda = \lambda_n = \frac{1 - n^2\,\pi^2}{L^2} \quad ; \quad n = 1, 2, 3, \cdots \,,$$

with corresponding eigenfunctions

$$X_n(x) = \sin\left(\frac{n\,\pi}{L} x\right) \quad ; \quad n = 1, 2, 3, \cdots \,.$$

(c) For choices of λ as given in (b), solve equation (3) to obtain the solutions

$$T_n(t) = \sin\left(\frac{c\,\sqrt{n^2\pi^2 - 1}}{L}\, t\right) \quad ; \quad n = 1, 2, 3, \ldots$$

(d) Using the above parts, write down the form of the solution $u(x,t)$ which satisfies the partial differential equation (1) along with the given homogeneous boundary and initial conditions. Your answer will depend on some number of yet to be determined constants.

(e) Finally, utilise the non-zero initial condition to find the constants in (d) in terms of the function $f(x)$, thereby determining the complete solution to the given initial-boundary value problem.

3. In this problem, we solve a system of ordinary differential equations using two different techniques adapted from our knowledge of solutions of ODEs. Consider the simultaneous ordinary differential equations

$$\frac{dx}{dt} = y + \sin t \,,$$

$$\frac{dy}{dt} = x + 2\,\cos t \,,$$

subject to the initial conditions $x(0) = 0$ and $y(0) = 0$.

(a) Laplace transforms approach: Let $X(s)$ and $Y(s)$ be the Laplace transforms of $x(t)$ and $y(t)$ respectively. By taking Laplace transforms of each of the above equations, obtain two simultaneous equations for $X(s)$ and $Y(s)$. Solve the above simultaneous equations to get

$$X(s) = \frac{3s}{(s^2+1)(s^2-1)} \,.$$

Determine $x(t)$ by inverting $X(s)$. Thereby, find $y(t)$.

(b) Second-order approach: Differentiate the first equation, and substitute for dy/dt from the second, in order to obtain a second-order constant-coefficient, non-homogeneous ODE for $x(t)$. Solve this using the standard methods (homogeneous and particular solutions), and thereby obtain $x(t)$. Thereafter, find $y(t)$ using any convenient method.