MATH2065: INTRO TO PDEs

Summer School 2012

Tutorial Solutions 2

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1. (a) Try $y_p = Ae^x$, where A is a constant to be determined. Differentiating this form gives

$$y_p' = Ae^x \quad \text{and} \quad y_p'' = Ae^x .$$

Substituting into the ODE we find $A = -\frac{1}{3}$. A particular solution is therefore

$$y_p = -\frac{1}{3}e^x .$$

(b) Try $y_p = A\,x + B$. Then, $y'' = 0$, and by substitution, we get

$$-9(A\,x + B) = x + 18 \qquad \Rightarrow \qquad -9A = 1 \;\text{ and }\; -9B = 18.$$

Thus, $A = -1/9$ and $B = -2$, yielding the particular solution

$$y_p(x) = -\frac{1}{9}x - 2.$$

(c) Set $y_p(x) = A\cos x + B\sin x$. Then, $y_p' = -A\sin x + B\cos x$ and $y_p'' = -A\cos x - B\sin x$. Upon substitution into the ODE we get

$$(-A\cos x - B\sin x) - (-A\sin x + B\cos x) + 2\,(A\cos x + B\sin x) = -2\sin x.$$

Equating coefficients of $\cos x$ and $\sin x$, we get the simultaneous equations $A - B = 0$ and $A + B = -2$. Solving, $A = B = -1$, and hence a particular solution is

$$y_p(x) = -\cos x - \sin x .$$

(d) Try $y_p = Ax^2 + Bx + C$, where A, B and C are constants (note that we must guess the most general form of a second-order polynomial). Differentiating this form gives

$$y_p' = 2Ax + B \quad \text{and} \quad y_p'' = 2A .$$

Substituting into the ODE and equating coefficients of powers of x gives $A = 1$, $B = 0$ and $C = -2$. A particular solution is therefore

$$y_p(x) = x^2 - 2.$$

(e) Note that e^{3x} is a solution of the homogeneous ODE. So substituting Ae^{3x} as a particular solution to the inhomogeneous equation will not work, since the left-hand side will produce a zero. Therefore, we try $y_p = Axe^{3x}$, where A is a constant, using the modification rule of multiplying the initial guess by a factor of x. Differentiating this form gives

$$y_p' = Ae^{3x}(1 + 3x) \quad \text{and} \quad y_p'' = Ae^{3x}(6 + 9x) .$$

Substituting into the ODE gives $A = \frac{1}{3}$. A particular solution is therefore

$$y_p = \frac{1}{3}xe^{3x} .$$

2. The ODEs are all second order, linear, inhomogeneous, with constant coefficients.

(a) First find y_h. The characteristic equation is $\lambda^2 + \lambda - 6 = 0 \Rightarrow \lambda = 2, -3$. The general solution of the homogeneous equation is therefore

$$y_h = C_1 e^{2x} + C_2 e^{-3x} .$$

For y_p, the form of $R(x)$ suggests we try $y_p = Ae^x$, where A is a constant. Differentiating this form gives

$$y_p' = Ae^x \quad \text{and} \quad y_p'' = Ae^x .$$

Substituting into the ODE gives $A = -\frac{5}{4}$. The general solution of the inhomogeneous ODE is therefore

$$y = y_h + y_p = C_1 e^{2x} + C_2 e^{-3x} - \frac{5}{4} e^x .$$

Now, using the initial conditions, $y(0) = 0 \Rightarrow C_1 + C_2 - \frac{5}{4} = 0$ and $y'(0) = 2 \Rightarrow 2C_1 - 3C_2 - \frac{5}{4} = 2$. Hence $C_1 = \frac{7}{5}$, $C_2 = -\frac{3}{20}$. Thus the particular solution which satisfies the initial conditions is

$$y(x) = \frac{7}{5} e^{2x} - \frac{3}{20} e^{-3x} - \frac{5}{4} e^x .$$

(b) First find y_h. The characteristic equation is $\lambda^2 + 3\lambda + 2 = 0 \Rightarrow \lambda = -2, -1$. The general solution of the homogeneous equation is therefore

$$y_h = C_1 e^{-2x} + C_2 e^{-x} .$$

For y_p, the form of $R(x)$ suggests we try $y_p = (Ax + B)e^{4x}$, where A and B are constants. Differentiating this form gives

$$y_p' = Ae^{4x} + 4(Ax + B)e^{4x} \quad \text{and} \quad y_p'' = 8Ae^{4x} + 16(Ax + B)e^{4x} .$$

Substituting into the ODE gives $A = \frac{1}{30}$, $B = -\frac{11}{900}$. The general solution of the inhomogeneous ODE is therefore

$$y = y_h + y_p = C_1 e^{-2x} + C_2 e^{-x} + (\frac{1}{30} x - \frac{11}{900}) e^{4x} .$$

Now, using the initial conditions, $y(0) = 0$, $y'(0) = 1 \quad \Rightarrow C_1 = -\frac{37}{36}$ and $C_2 = \frac{26}{25}$. Thus the particular solution which satisfies the initial conditions is

$$y(x) = -\frac{37}{36} e^{-2x} + \frac{26}{25} e^{-x} + (\frac{1}{30} x - \frac{11}{900}) e^{4x} .$$

(c) First find y_h. The characteristic equation is $\lambda^2 - 4\lambda + 3 = 0 \Rightarrow \lambda = 1, 3$. The general solution of the homogeneous equation is therefore

$$y_h = C_1 e^{3x} + C_2 e^x .$$

For y_p the form of $R(x)$ suggests we try $y_p = A \sin x + B \cos x$, where A and B are constants. Differentiating and the substituting into the ODE gives $A = \frac{1}{5}$, $B = \frac{2}{5}$. The general solution of the inhomogeneous ODE is therefore

$$y = y_h + y_p = C_1 e^{3x} + C_2 e^x + \frac{1}{5} \sin x + \frac{2}{5} \cos x .$$

Now, using the initial conditions, $y(0) = 1$, $y'(0) = 0 \quad \Rightarrow C_1 = -\frac{2}{5}$ and $C_2 = 1$. Thus the particular solution which satisfies the initial conditions is

$$y(x) = -\frac{2}{5} e^{3x} + e^x + \frac{1}{5} \sin x + \frac{2}{5} \cos x .$$

3. We know $y_1'' + ay_1' + by_1 = f_1$, and also $y_2'' + ay_2' + by_2 = f_2$. Simply adding the two gives

$$(y_1'' + y_2'') + a(y_1' + y_2') + b(y_1 + y_2) = f_1 + f_2\,,$$

thus demonstrating that $y_1 + y_2$ satisfies the given ODE.

(a) Let $f_1 = x$, and look for a corresponding particular solution y_1 of the form $y_1 = Ax+B$, where A and B are constants. Differentiating and substituting, we find $A = \frac{1}{3}, \quad B = -\frac{10}{9}$.
Similarly for $f_2 = \cos x$, try $y_2 = C\sin x + D\cos x$, where C and D are constants. Differentiating and substituting, we find $C = \frac{1}{10}, \quad D = 0$. Hence

$$y_p = \frac{1}{3}x - \frac{10}{9} + \frac{1}{10}\sin x\,.$$

(b) Let $f_1 = 2e^x$, and try $y_1 = Ae^x$, where A is a constant. Differentiating and substituting, we find $A = \frac{1}{6}$.
For $f_2 = \sin x$, try $y_2 = B\sin x + C\cos x$, where B and C are constants. Differentiating and substituting, we find $B = \frac{1}{10}, \quad C = -\frac{1}{10}$. Hence

$$y_p = \frac{1}{6}e^x + \frac{1}{10}\sin x - \frac{1}{10}\cos x\,.$$

(c) Let $f_1 = e^x$. Since e^x is a solution of the homogeneous equation, try $y_1 = Axe^x$, where A is a constant. Differentiating and substituting gives $A = -\frac{1}{2}$.
Similarly for $f_2 = e^{2x}$, try $y_2 = Ce^{2x}$, where C is a constant. Differentiating and substituting gives $C = -1$. Hence

$$y_p = -\frac{1}{2}xe^x - e^{2x}\,.$$

4. The governing differential equation is

$$\ddot{x} + 4\,x = \cos\omega t\,.$$

(a) The homogeneous solution x_h obeys $\ddot{x} + 4x = 0$. This has characteristic equation $\lambda^2 + 4 = 0$, and hence $\lambda = \pm 2i$. The corresponding solutions are therefore $\cos 2t$ and $\sin 2t$, leading to

$$x_h(t) = C_1\,\cos(2t) + C_2\,\sin(2t)\,,$$

where C_1 and C_2 are arbitrary constants. Note that this is an oscillatory function – it keeps on periodically wiggling.

(b) To find $x_p(t)$, we use undetermined coefficients. Based on the form of the inhomogeneity, the correct guess is

$$x_p(t) = A\,\cos(\omega t) + B\,\sin(\omega t)\,. \tag{1}$$

However, we note that there is a possible problem: if $\omega = 2$, this is actually part of the homogeneous solution, and therefore will not work (upon substitution into the ODE, the left-hand side will yield zero, giving us no information on the undetermined coefficients A and B). Thus, the above guess is valid *only if* $\omega \neq 2$. Under this condition, substitution gives

$$\left(-A\omega^2\cos\omega t - B\omega^2\sin\omega t\right) + 4\left(A\cos\omega t + B\sin\omega t\right) = \cos\omega t\,.$$

Equating coefficients of $\sin \omega t$, we get $B = 0$. Equating coefficients of $\cos \omega t$, we find that $A = 1/(4 - \omega^2)$, and so

$$x_p(t) = \frac{1}{4 - \omega^2} \cos(\omega t) \qquad (\omega \neq 2)\,.$$

If $\omega = 2$, the above does not work. We might have guessed (1) with $\omega = 2$, but since this is a part of the homogeneous solution, we need to guess t times that; i.e.,

$$x_p(t) = A\,t\,\cos(2t) + B\,t\,\sin(2t)\ .$$

Substituting into $\ddot{x} + 4x = \cos 2t$ (since $\omega = 2$ now), we get, after some algebra (terms with multiplicative factors of t thankfully cancel, as they must),

$$-4\,A \sin(2t) + 4\,B \cos(2t) = \cos(2t)\ .$$

Equating coefficients as before, we get $A = 0$ and $B = 1/4$. Thus, if $\omega = 2$, then

$$x_p(t) = \frac{t}{4} \sin(2t) \qquad (\omega = 2)\,.$$

(c) Since $x(t) = x_h(t) + x_p(t)$, we have

$$x(t) = \begin{cases} C_1 \cos(2t) + C_2 \sin(2t) + \dfrac{1}{4 - \omega^2} \cos(\omega t) & (\text{if } \omega \neq 2) \\ C_1 \cos(2t) + C_2 \sin(2t) + \dfrac{t}{4} \sin(2t) & (\text{if } \omega = 2) \end{cases}$$

If $\omega \neq 2$, the solution is oscillatory, and so the block just keeps on periodically repeating its motion (it goes back and forth indefinitely – this is reasonable in that we have set $\gamma = 0$ here, and hence there is no friction causing it to slow down). If $\omega = 2$, on the other hand, the term $t \sin 2t$ appears in the solution. This grows (since t does), while oscillating as well. In fact, it grows without bound, oscillating between the lines $x = \pm\, t$. The block will therefore start oscillating wildly.

This phenomenon is called *resonance*. If the block is excited (forced) at a frequency which is equal to its natural frequency (the frequency at which it would oscillate if left to its own devices), the oscillations grow wilder. In this case, the resonance frequency is 2.