

MATH2065: INTRO TO PDEs

Summer School 2012

Tutorial Solutions 3

1. (a) $\frac{6}{(s+2)^4}$ (each time we multiply by $-t$, it is equivalent to differentiating the Laplace transform with respect to s)
- (b) $\frac{8s}{(s^2+16)^2}$ (same reason as above)
- (c) $\frac{e^{-3s}}{s}$ (directly from the table)
- (d) $\frac{4}{(s-3)^2+16}$ (take the Laplace transform of $\sin 4t$, and then use the shifting in s property)
- (e) $\frac{(s+4)^2-36}{[(s+4)^2+36]^2}$ (first, use the multiplication by $-t$ property to compute the Laplace transform of $t \cos 6t$, and then use the shifting in s property on that to deal with the exponential multiplier e^{-4t})
- (f) $e^{-5s} \left(\frac{2}{s^3} + \frac{10}{s^2} + \frac{25}{s} \right) - e^{-8s} \left(\frac{2}{s^3} + \frac{16}{s^2} + \frac{64}{s} \right)$ (one can do this using the shifting in t property in conjunction with the Heaviside function, but it is probably easier to do this directly using the definition of the Laplace transform, leading to the expression

$$\mathcal{L}\{f(t)\}(s) = \int_5^8 t^2 e^{-st} dt,$$

which can then be evaluated by integrating twice by parts)

- (g) $\frac{24e^{-s}}{s^5}$ (use the shifting in t property)
2. (a) $\frac{1}{2} \sin 2t$ (directly from the table)
 - (b) $\frac{1}{2} H(t-3) \sin [2(t-3)]$ (from the above in conjunction with the t -shifting property)
 - (c) $\frac{1}{6!} t^6 e^{4t}$ (first, note that the inverse Laplace transform of $1/s^7$ is $t^6/6!$ by the property of the Laplace transform of t^n , and then apply the s -shift property)
 - (d) Since $s^2 + 8s + 7 = (s+1)(s+7)$, employ partial fractions

$$\frac{s}{s^2 + 8s + 7} = \frac{A}{s+1} + \frac{B}{s+7}$$

and hence $s = A(s+7) + B(s+1)$. One can equate coefficients of s^0 and s^1 to obtain two simultaneous equations to solve for A and B , or simply substitute $s = -7$ (which tells us that $B = 7/6$) and $s = -1$ (from which $A = -1/6$). Thus

$$\mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 8s + 7} \right\} = -\frac{1}{6} e^{-t} + \frac{7}{6} e^{-7t}.$$

- (e) Since the quadratics which appear in the denominator are not factorisable, the correct partial fraction guess is (see your first-year calculus, or precalculus, notes for a review of partial fractions if needed):

$$\frac{s}{(s^2 + 1)(s^2 + 4)} = \frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + 4},$$

which leads to the expression

$$s = (As + B)(s^2 + 4) + (Cs + D)(s^2 + 1) = (A + C)s^3 + (B + D)s^2 + (4A + C)s + (4B + D).$$

Equating coefficients of s^3 , s^2 , s^1 and s^0 respectively,

$$A + C = 0 \quad , \quad B + D = 0 \quad , \quad 4A + C = 1 \quad , \quad 4B + D = 0.$$

Solving these simultaneously, $A = 1/3$, $B = 0$, $C = -1/3$, $D = 0$. Thus,

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{s}{(s^2 + 1)(s^2 + 4)} \right\} &= \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 1} \right\} - \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 4} \right\} \\ &= \frac{1}{3} \cos t - \frac{1}{3} \cos 2t. \end{aligned}$$

- (f) We will first ignore the exponential term, which we will deal with later using properties of Laplace transforms. Now since $s^3 + 9s = s(s^2 + 9)$, the appropriate partial fraction decomposition is (recall that the guess for the numerator should in general be one power less than that of the denominator)

$$\frac{s + 2}{s^3 + 9s} = \frac{A}{s} + \frac{Bs + C}{s^2 + 9}$$

and hence

$$s + 2 = A(s^2 + 9) + s(Bs + C).$$

Equating coefficients of s^2 , s^1 and s^0 respectively,

$$A + B = 0 \quad , \quad C = 1 \quad , \quad 9A = 2,$$

and hence $A = 2/9$, $B = -2/9$, $C = 1$. Therefore

$$\mathcal{L}^{-1} \left\{ \frac{s + 2}{s^3 + 9s} \right\} = \frac{2}{9} - \frac{2}{9} \cos 3t + \frac{1}{3} \sin 3t.$$

Now

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{s + 2}{s^3 + 9s} (1 - 5e^{-4s}) \right\} &= \mathcal{L}^{-1} \left\{ \frac{s + 2}{s^3 + 9s} \right\} - 5 \mathcal{L}^{-1} \left\{ \frac{s + 2}{s^3 + 9s} e^{-4s} \right\} \\ &= \frac{2}{9} - \frac{2}{9} \cos 3t + \frac{1}{3} \sin 3t - 5H(t - 4) \left(\frac{2}{9} - \frac{2}{9} \cos [3(t - 4)] + \frac{1}{3} \sin [3(t - 4)] \right), \end{aligned}$$

by utilising the t -shifting property.

3. (a) From the property which we have already shown, multiplication by $-t$ in the t -domain is equivalent to taking a derivative in the s -domain. Simply do this n times, and we get the result we want. (Interested students are invited to prove this more rigorously using mathematical induction.)

(b) Use the property in (a), and choose $f(t) = 1$. Then $F(s) = 1/s = s^{-1}$. Thus,

$$\mathcal{L}\{(-t)^n\} = \frac{d^n}{ds^n} s^{-1} = (-1)(-2)(-3)\cdots(-n)s^{-n-1} = \frac{(-1)^n n!}{s^{n+1}}.$$

Multiplying both sides by $(-1)^n$ gives the required result.

(c) We know that $\mathcal{L}\{g'(t)\} = sG(s) - g(0)$ for any function g . Take $g(t) = f'(t)$. Then

$$\begin{aligned}\mathcal{L}\{f''(t)\} = \mathcal{L}\{g'(t)\} &= sG(s) - g(0) \\ &= s\mathcal{L}\{f'(t)\} - f'(0) \\ &= s(sF(s) - f(0)) - f'(0) \\ &= s^2F(s) - sf(0) - f'(0).\end{aligned}$$

One can similarly obtain, for example, that

$$\mathcal{L}\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0).$$

4. Applying the Laplace transform to the ODE, with the understanding that $Y(s) = \mathcal{L}\{y(t)\}$, we get

$$(sY(s) - y(0)) + Y(s) = \frac{1}{s}$$

in which it is known that $y(0) = 2$. Therefore,

$$Y(s) = \frac{1}{s+1} \left(\frac{1}{s} + 2 \right) = \frac{1}{s} + \frac{1}{s+1}$$

by utilising partial fractions and a little algebra. Thus

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 1 + e^{-t}.$$

5. By taking the Laplace transform of the differential equation, we obtain

$$(s^2Y(s) - sy(0) - y'(0)) + 4Y(s) = \frac{1}{s^2 + 1}$$

and since $y(0) = y'(0) = 0$, we get

$$Y(s) = \frac{1}{(s^2 + 1)(s^2 + 4)}.$$

Utilising partial fractions in the form

$$\frac{1}{(s^2 + 1)(s^2 + 4)} = \frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + 4}$$

we see that

$$1 = (A + C)s^3 + (B + D)s^2 + (4A + C)s + (4B + D).$$

Equating coefficients and solving the four equations simultaneously, we get $B = 1/3$, $D = -1/3$, and $A = C = 0$. Thus,

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{3} \sin t - \frac{1}{6} \sin 2t.$$

Note that this can also be solved using the standard techniques which we have discussed before (finding the homogeneous solution, and then a particular solution using undetermined coefficients, adding these to form the general solution, and then determining the arbitrary constants). However, this problem asks that you solve it using Laplace transforms. As an exercise, you can verify that the other approach gives the same result.