MATH2065: INTRO TO PDEs

Summer School 2012

Tutorial Solutions 4

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1. (a) Notice that the voltage input is zero for $0 < t < a$, is V_0 between a and b, and is zero thereafter – so this *is* a square pulse. Denoting $I(s) = \mathcal{L}\{i(t)\}$, and applying the Laplace transform to the differential equation, we get

$$L[sI(s) - i(0)] + R\,I(s) = V_0 \left[\frac{e^{-as}}{s} - \frac{e^{-bs}}{s}\right],$$

from which (since $i(0) = 0$), we obtain

$$I(s) = \frac{V_0}{s\,(Ls + R)}\left(e^{-as} - e^{-bs}\right).$$

Now, it is necessary to invert the Laplace transform. Do not get confused with all the parameters – remember that everything except s is a constant. The term we need to be able to invert is $1/\left(s(Ls + R)\right)$. So we find its partial fraction decomposition

$$\frac{1}{s\,(Ls + R)} = \frac{1}{R}\left(\frac{1}{s} - \frac{L}{Ls + R}\right) = \frac{1}{R}\left[\frac{1}{s} - \frac{1}{s + R/L}\right].$$

Hence,

$$I(s) = \frac{V_0}{R}\left[\frac{1}{s}\left(e^{-as} - e^{-bs}\right) - \frac{1}{s + R/L}\left(e^{-as} - e^{-bs}\right)\right].$$

Now, we note that

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1 \qquad \text{and} \qquad \mathcal{L}^{-1}\left\{\frac{1}{s + R/L}\right\} = e^{-Rt/L},$$

and hence we can invert each term to get the final solution:

$$i(t) = \frac{V_0}{R}\left\{[H(t-a) - H(t-b)] - \left[H(t-a)e^{-R(t-a)/L} - H(t-b)e^{-R(t-b)/L}\right]\right\}.$$

(b) If $V(t) = V_0\delta(t - a)$ instead, its Laplace transform is V_0e^{-as}. This needs to replace the right-hand side of the first equation in part (a), leading to

$$I(s) = \frac{V_0}{Ls + R}e^{-as} = \left(\frac{V_0}{L}\right)\frac{1}{s + R/L}e^{-as}.$$

If the exponential term was absent, the inverse Laplace transform would be $(V_0/L)e^{-Rt/L}$. The exponential term can be handled using the shifting property of Laplace transforms, and hence the solution is

$$i(t) = \frac{V_0}{L}H(t-a)e^{-R(t-a)/L}.$$

The current therefore remains zero in the circuit until the time of the spike, at which point it jumps to V_0/L, decaying exponentially thereafter.
<u>Some comments:</u> The advantage in using Laplace transforms rather than attempting to solve the ODE using our earlier methods is that the solution can be written using

one expression. Using standard methods for (a), we would have had to separately solve for the various domains $0 < t < a$, $a < t < b$ and $t > b$, and have to worry about the 'initial' conditions necessary for each such domain. For example, we would need to determine $i(a)$ as an initial condition for the second domain, based on our solution for the first domain. Laplace transforms takes care of all this automatically. Our other standard methods (undetermined coefficients) do not work at all when the discontinuity is much worse, as occurs in the spike example for part (b). Laplace transforms are incredibly powerful when we have *discontinuous* functions in the ODE.

2. Denote the right-hand side (the inhomogeneity) by $f(t)$, and its Laplace transform by $F(s)$. We can either write it in terms of Heaviside functions (switching the functions $4e^{-t}$ and 2 "on" and "off" as needed), or can compute $F(s)$ directly using the definition of the Laplace transform:

$$\begin{aligned} F(s) &= \int_0^\infty f(t)e^{-st}\,dt \\ &= \int_0^8 4e^{-t}e^{-st}\,dt + \int_8^\infty 2e^{-st}\,dt \\ &= 4\left.\frac{e^{-(s+1)t}}{-(s+1)}\right]_0^8 + 2\left.\frac{e^{-st}}{-s}\right]_8^\infty \\ &= \frac{4}{s+1}\left[1-e^{-8(s+1)}\right] + \frac{2}{s}e^{-8s}\,. \end{aligned}$$

Therefore, when applying the Laplace transform to the ODE, we get

$$sY(s) - y(0) + 3Y(s) = \frac{4}{s+1}\left[1-e^{-8(s+1)}\right] + \frac{2}{s}e^{-8s}\,,$$

where $Y(s)$ is the Laplace transform of the solution $y(t)$. Putting in $y(0) = 1$ and solving for $Y(s)$,

$$Y(s) = \frac{4}{(s+1)(s+3)}\left[1-e^{-8}e^{-8s}\right] + \frac{2}{s(s+3)}e^{-8s} + \frac{1}{s+3}\,.$$

Now, using partial fractions, we see that

$$\mathcal{L}^{-1}\left\{\frac{4}{(s+1)(s+3)}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s+1} - \frac{2}{s+3}\right\} = 2e^{-t} - 2e^{-3t}$$

and

$$\mathcal{L}^{-1}\left\{\frac{2}{s(s+3)}\right\} = \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+3}\right\} = \frac{2}{3}(1-e^{-3t})\,.$$

Therefore, using the t-shift property as well,

$$y(t) = 2e^{-t} - 2e^{-3t} - e^{-8}H(t-8)\left(2e^{-(t-8)} - 2e^{-3(t-8)}\right) + \frac{2}{3}H(t-8)\left(1-e^{-3(t-8)}\right) + e^{-3t}\,.$$

3. What we are asked to show is the following:

$$\int_0^t f(t-\bar{t})\,g(\bar{t})\,d\bar{t} = \int_0^t g(t-\bar{t})\,f(\bar{t})\,d\bar{t}\,, \tag{1}$$

that is, that it does not matter in which order you choose the two functions. Let us start with the integral on the left-hand side above. Note that t is a constant as far as this integral is concerned, and the integration is over the dummy variable $\bar{t}$. Change the integration variable from $\bar{t}$ to w, where w is defined by

$$w = t - \bar{t}\,.$$

Now, when $\bar{t} = 0$, $w = t$. When $\bar{t} = t$, $w = 0$. Moreover, $dw = -d\bar{t}$. Therefore,

$$\int_0^t f(t-\bar{t})\, g(\bar{t})\, d\bar{t} = \int_t^0 f(w)\, g(t-w)\ (-dw) = \int_0^t g(t-w)\, f(w)\, dw\,,$$

where the negative sign has been used to switch the limits of integration. This expression is *exactly* the same as that on the right-hand side of (1), since the w (and the $\bar{t}$) are dummy variables. That is, they actually are not present in the final expression, they are just placeholders to indicate that the integrand is to be evaluated at all values ranging from 0 to t.

4. We first note that

$$\frac{1}{(s^2+1)^2} = \left(\frac{1}{s^2+1}\right)\left(\frac{1}{s^2+1}\right).$$

Now, since each of these terms is equal to $\mathcal{L}\{\sin t\}$, it follows from the convolution theorem that

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} &= \sin t \star \sin t \\
&= \int_0^t \sin(t-\bar{t})\, \sin\bar{t}\, d\bar{t} \\
&= \frac{1}{2}\int_0^t \left[\cos(2\bar{t}-t) - \cos t\right] d\bar{t} \\
&= \frac{1}{2}\left[\frac{\sin(2\bar{t}-t)}{2}\right]_{\bar{t}=0}^{t} - \frac{1}{2}\cos t \int_0^t d\bar{t} \\
&= \frac{1}{2}\left[\frac{\sin t}{2} - \frac{\sin(-t)}{2}\right] - \frac{1}{2} t\, \cos t \\
&= \frac{\sin t - t\, \cos t}{2}\,.
\end{aligned}$$

In the above, the given trigonometric identity has been used to simplify the first integral, and the fact that the sine function is odd (that is, $\sin(-x) = -\sin x$) has been used towards the end.

5. Let $Y(s) = \mathcal{L}\{y(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$. Taking the Laplace transform of both sides of the given differential equation, and utilising the initial conditions, gives

$$s^2\, Y(s) - s - 1 - Y(s) = G(s)\,.$$

Solving for $Y(s)$, we have

$$Y(s) = \frac{s+1}{s^2-1} + \left(\frac{1}{s^2-1}\right) G(s) = \frac{1}{s-1} + \left(\frac{1}{s^2-1}\right) G(s)\,.$$

Hence

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\, G(s)\right\} \\
&= e^t + \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\, G(s)\right\}.
\end{aligned}$$

We know (from the standard Laplace transforms table) that

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = \sinh t\,,$$

and so we can express

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\,G(s)\right\} = \sinh t \star g(t)$$

by using the convolution property. Therefore,

$$y(t) = e^t + \int_0^t \sinh(t-\bar{t})\,g(\bar{t})\,d\bar{t}\,.$$

This is a very powerful method of solution, in that the above indicates the solution for *any* given inhomogeneity $g(t)$. Of course, doing the actual integration above may not be easy for complicated g. Note that we may also have written the integral in the "opposite" way, since $f \star g = g \star f$, that is, in the alternative form

$$y(t) = e^t + \int_0^t g(t-\bar{t})\,\sinh(\bar{t})\,d\bar{t}\,,$$

which is also correct.

6. Notice that

$$\int_0^t f(\bar{t})\,d\bar{t} = \int_0^t g(t-\bar{t})\,f(\bar{t})\,d\bar{t}$$

for the choice of the function $g(t) = 1$. By the convolution property in the Laplace transform table, the Laplace transform of the above is $F(s)\,G(s)$. But since $g(t) = 1$, $G(s) = 1/s$. Therefore, the required Laplace transform is

$$\mathcal{L}\left\{\int_0^t f(\bar{t})\,d\bar{t}\right\} = \frac{F(s)}{s}$$

where $F(s)$ is the Laplace transform of $f(t)$.

7. (a) This corresponds to choosing $a = -1$, $b = 3$, and $f(t) = t\cos 3t$ in the definition of the Dirac delta function. Therefore, if $b \in (-1, 3)$,

$$\int_{-1}^{3} \delta(t-b)\,t\,\cos 2t\,\mathrm{d}t = t\cos 2t\Big|_{t=b} = b\,\cos 2b\,.$$

On the other hand, if $b < -1$ or if $b > 3$, the Dirac delta function contributes zero to the integrand, since the integral is over $(-1, 3)$. Therefore, the integral yields zero. The information we have now is that

$$\int_{-1}^{3} \delta(t-b)\,t\,\cos 2t\,\mathrm{d}t = \begin{cases} b\,\cos 2b & \text{if } b \in (-1,3)\,, \\ 0 & \text{if } b < -1 \text{ or } b > 3\,. \end{cases}$$

Unfortunately, it is not possible to define the value of the integral if b is exactly on one of the endpoints -1 or 3 (this may not be at all obvious – but the definition of the Dirac delta function does not cover this case, which turns out to be ill-defined). A clever way of rewriting the solution above is

$$\int_{-1}^{3} \delta(t-b)\,t\,\cos 2t\,\mathrm{d}t = [H(b+1) - H(b-3)]\;b\,\cos 2b\,.$$

(convince yourself that this works). This also has the advantage of not giving any well-defined value at the problematic points $b = -1$ and $b = 3$, since the unit-step functions $H(b+1)$ and $H(b-3)$ are respectively ill-defined at those values.

(b) Applying the definition of the Laplace transform,

$$\begin{aligned}
\mathcal{L}\left\{\delta(t-b)f(t)\right\} &= \int_0^\infty \delta\left(t-b\right)\, f(t)\, e^{-st}\, \mathrm{d}t \\
&= f(t)\, e^{-st}\Big|_{t=b} \\
&= f(b)\, e^{-sb}\,.
\end{aligned}$$

However, we note that the above only works if b is within the inteval $(0, \infty)$, since if not, the Dirac delta function contributes nothing to the integral, which is just zero. This can be compactly stated as

$$\mathcal{L}\left\{\delta(t-b)f(t)\right\} = H(b)\, f(b)\, e^{-sb}\,,$$

since the unit-step function $H(b)$ will be zero if $b < 0$. Note that once again, it is not clear what the value would be if $b = 0$ (this is not defined, just as the unit-step function $H(b)$ is not defined when $b = 0$).