MATH2065: INTRO TO PDEs

Summer School 2011

Tutorial Questions 5

1. To practice your partial differentiation, verify that

$$u(x,t) = \sin(2\pi x)\, e^{-4\pi^2 kt}$$

is a solution to the heat equation

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}\,.$$

2. Verify that $u(x,t) = \sin(x - ct)$ is a solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\,\frac{\partial^2 u}{\partial x^2}\,,$$

which one might also write using the subscript notation for partial derivatives as

$$u_{tt} = c^2\, u_{xx}\,.$$

Indeed, if f and g are *any* two differentiable functions, verify that

$$u(x,t) = f(x - ct) + g(x + ct)$$

works as a solution to the wave equation. (This rather amazing fact is called *D'Alembert's solution* to the wave equation.)

3. Find the equilibrium temperature distribution for a one-dimensional homogeneous rod with no heat source and boundary conditions $u(0) = T$ and $u(L) = 0$.

4. Find the equilibrium temperature distribution for a one-dimensional homogeneous rod with heat source $Q = K_0 x^2$ and boundary conditions $u(0) = T$ and $\dfrac{\partial u}{\partial x}(L) = 0$. The heat equation in this more general case with heat source is (K_0, c, ρ are positive constants)

$$c\,\rho\,\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\,\frac{\partial u}{\partial x}\right) + Q\,.$$

5. Verify that Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

satisfies the "principle of superposition"; that is, that any linear combination of solutions is itself a solution.

6. Apply the method of separation of variables tor the following three PDEs and thus find the resulting ODEs. (Note that you are not asked to *solve* any of the ODEs that you get.)

(a) $\dfrac{\partial u}{\partial t} = \dfrac{k}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right)$

(b) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

(c) $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^4 u}{\partial x^4}$