MATH2065: INTRO TO PDEs

Summer School 2012

Tutorial Solutions 5

1. Taking partial derivatives of u:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= -4\pi^2 k \sin 2\pi x e^{-4\pi^2 kt}\,, \\
\frac{\partial u}{\partial x} &= 2\pi \cos 2\pi x e^{-4\pi^2 kt}\,, \\
\frac{\partial^2 u}{\partial x^2} &= -4\pi^2 \sin 2\pi x e^{-4\pi^2 kt}\,.
\end{aligned}$$

The last of these when multiplied by k gives $\frac{\partial u}{\partial t}$ as required.

2. Let $u(x,t) = \sin(x - ct)$. Taking partial derivatives, and utilising the chain rule,

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \cos(x-ct)\,(-c) = -c\,\cos(x-ct)\,, \\
\frac{\partial^2 u}{\partial t^2} &= (-c)\,(-\sin(x-ct))\,(-c) = -c^2 \sin(x-ct)\,. \\
\frac{\partial u}{\partial x} &= \cos(x-ct)\,, \\
\frac{\partial^2 u}{\partial x^2} &= -\sin(x-ct)\,.
\end{aligned}$$

Therefore, multiplying u_{xx} by c^2 gives us u_{tt}, which means that

$$u_{tt} = c^2\, u_{xx}\,.$$

Now consider the purported D'Alembert's solution

$$u(x,t) = f(x-ct) + g(x+ct)\,.$$

Note that each of f and g are functions of *one* variable; what appears above is that the combination $x - ct$ has been fed into f, and the combination $x + ct$ into g. Now, using the chain rule,

$$\begin{aligned}
\frac{\partial u}{\partial x} &= f'(x-ct) + g'(x+ct)\,, \\
\frac{\partial^2 u}{\partial x^2} &= f''(x-ct) + g''(x+ct)\,. \\
\frac{\partial u}{\partial t} &= f'(x-ct)(-c) + g'(x+ct)(+c) = c\,(-f'(x-ct) + g'(x+ct))\,, \\
\frac{\partial^2 u}{\partial t^2} &= c\,(-f''(x-ct)(-c) + g''(x+ct)(+c)) = c^2\,(f''(x-ct) + g''(x+ct))\,.
\end{aligned}$$

and hence $u_{tt} = c^2 u_{xx}$. When applying the chain rule above, what we have done in each instance is given by the following example:

$$\begin{aligned}
\frac{\partial}{\partial t} f\,(x-ct) &= (\text{Derivative of}\,f\,\text{with respect to whatever's inside}) \\
&\quad \times (\text{Derivative of what's inside with respect to}\,t) \\
&= f'(x-ct)\frac{\partial}{\partial t}(x-ct) = f'(x-ct)\,(-c)\,.
\end{aligned}$$

We might also do this problem as follows. Let $\xi(x,t) = x - ct$ and $\eta(x,t) = x + ct$. Then we have

$$u(x,t) = f(\xi(x,t)) + g(\eta(x,t)) \,.$$

Now, a change in t causes changes in u through two processes, which we must sum over: (i) a change in ξ, which causes a change in f, and (ii) a change in η, which causes a change in g. Therefore, the chain rule applied to taking the derivative of u with respect to t gives us

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{df}{d\xi}\frac{\partial \xi}{\partial t} + \frac{dg}{d\eta}\frac{\partial \eta}{\partial t} \\ &= f'(\xi)(-c) + g'(\eta)(+c) = -cf'(x-ct) + cg'(x+ct)\,.\end{aligned}$$

Taking the other derivatives is a continuation of this process, and what has been presented in the first solution to this problem was essentially a short cut. Going through this process in a step-by-step fashion would yield the same result.

3. In equilibrium, since u will not change with respect to t, it must satisfy

$$\frac{d^2u}{dx^2} = 0$$

by tossing out the partial t-derivative in the heat equation. Integrating this twice (note: this is now an ODE), we get

$$u(x) = Ax + B$$

for some constants A and B. (Interested students are recommended to also obtain this in a slightly different way – by considering the ODE as a constant-coefficient, second-order homogeneous ODE, and using the method of the characteristic equation.) Since we need $u(0) = T$, we have the condition $B = T$. Since we also need $u(L) = 0$, we get the condition $AL + B = 0$. Solving for A and B (our unknowns), we get $A = -T/L$ and $B = T$. Hence, the equilibrium temperature distribution is

$$u(x) = -\frac{T}{L}\,x + T\,.$$

4. The Q here represents heat sources, and the heat equation in this more general case is

$$c\,\rho\,\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\,\frac{\partial u}{\partial x}\right) + Q\,.$$

At equilibrium, there is no variation in t, and thus

$$0 = \frac{d}{dx}\left(K_0\,\frac{du}{dx}\right) + Q\,.$$

Now substitute $Q = K_0 x^2$ as given in this problem, where K_0 be a constant. Thus we get the equation

$$\frac{d^2u}{dx^2} = -x^2\,. \tag{1}$$

Integrating this once

$$\frac{du}{dx} = -\frac{1}{3}x^3 + A\,,$$

where A is an integration constant. Integrating again,

$$u = -\frac{x^4}{12} + Ax + B$$

where B is another constant. Alternatively, one can determine the general solution to (1) by using the ideas of constant-coefficient, second-order inhomogeneous equations. In any case, we now need to satisfy the condition $u(0) = T$. This gives $B = T$. Secondly, we need $u'(L) = 0$. This gives $A = L^3/3$. Thus the equilibrium temperature distribution in this case is

$$u(x) = -\frac{x^4}{12} + \frac{L^3\,x}{3} + T\,.$$

5. Let $u_1(x,y)$ and $u_2(x,y)$ each be solutions to Laplace's equation as stated. Now consider

$$U(x,y) = C_1 u_1(x,y) + C_2 u_2(x,y)$$

for arbitrary constants C_1 and C_2. Now,

$$\begin{aligned}
\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} &= C_1\frac{\partial^2 u_1}{\partial x^2} + C_2\frac{\partial^2 u_2}{\partial x^2} + \left(C_1\frac{\partial^2 u_1}{\partial y^2} + C_2\frac{\partial^2 u_2}{\partial y^2}\right) \\
&= C_1\left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2}\right) + C_2\left(\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2}\right) \\
&= 0\,.
\end{aligned}$$

where each of the quantities in parentheses in the penultimate line above are zero since each of $u_1(x,t)$ and $u_2(x,t)$ satisfies Laplace's equation.

6. (a) Setting $u(r,t) = \phi(r)\,h(t)$ yields

$$\phi(r)\frac{dh}{dt} = \frac{k\,h}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)\,.$$

Dividing by $k\,\phi(r)\,h(t)$ yields

$$\frac{1}{k\,h}\,\frac{dh}{dt} = \frac{1}{r\,\phi(r)}\,\frac{d}{dr}\left(r\,\frac{d\phi}{dr}\right) = -\lambda$$

where λ is a separation constant, which must be independent of both r and t. This gives the two ODEs

$$\frac{dh}{dt} = -\lambda\,k\,h \qquad \text{and} \qquad \frac{1}{r}\,\frac{d}{dr}\left(r\,\frac{d\phi}{dr}\right) = -\lambda\,\phi\,.$$

This is of course alternatively written as

$$h'(t) = -\lambda\,k\,h(t) \qquad \text{and} \qquad \frac{1}{r}\,(r\,\phi'(r))' = -\lambda\,\phi(r)\,.$$

Note that the r-equation is nasty – it is not constant-coefficient, since the independent variable r appears throughout. Thankfully, we are not asked to solve it!

(b) Set $u(x,y) = \phi(x)\,h(y)$, to give

$$h\,\frac{d^2\phi}{dx^2} + \phi\,\frac{d^2h}{dy^2} = 0\,.$$

Dividing by $\phi(x)\,h(y)$ yields

$$\frac{1}{\phi}\,\frac{d^2\phi}{dx^2} = -\frac{1}{h}\,\frac{d^2h}{dy^2} = -\lambda\,,$$

which separates out to the ODEs

$$\frac{d^2\phi}{dx^2} = -\lambda\,\phi \qquad \text{and} \qquad \frac{d^2h}{dy^2} = \lambda\,h\,.$$

Of course, you may have decided to write these in the form

$$\phi''(x) = -\lambda\,\phi(x) \qquad \text{and} \qquad h''(y) = \lambda\,h(y)\,,$$

which would be equivalent.

(c) Set $u(x,t) = \phi(x)\,h(t)$, resulting in

$$\phi(x)\,h'(t) = k\,h(t)\,\phi''''(x)\,.$$

Dividing by $k\,\phi(x)\,h(t)$,

$$\frac{1}{k\,h(t)}\,h'(t) = \frac{1}{\phi(x)}\,\phi''''(x) = \lambda\,.$$

This separates into the two ODEs

$$h'(t) = \lambda\,k\,h(t) \qquad \text{and} \qquad \phi''''(x) = \lambda\,\phi(x)\,.$$

Note that you are not asked to *solve* any of the ODEs you get, which might be difficult – and in any case, you are not given any boundary/initial conditions, and so it cannot be done.