# MATH2065: INTRO TO PDEs

*Summer School 2012*

## Tutorial Questions 6

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1. Determine the eigenvalues $\lambda$ and corresponding eigenfunctions of the differential equation

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0$$

with boundary conditions $\phi(0) = 0$ and $\dfrac{d\phi}{dx}(L) = 0$. Analyze the three cases with real $\lambda$: $\lambda > 0$, $\lambda = 0$, $\lambda < 0$.

2. Solve the heat equation $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$ with boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$ and initial conditions $u(x,0) = 3\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}$.

3. As the previous problem but with initial condition $u(x,0) = \begin{cases} 1 & 0 < x \le L/2 \\ 2 & L/2 < x < L \end{cases}$.

4. Solve the heat equation $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$ with boundary conditions $\frac{\partial u}{\partial x}(0,t) = 0$ and $\frac{\partial u}{\partial x}(L,t) = 0$ for $t > 0$ and three different initial conditions given by

   (a) $u(x,0) = \begin{cases} 0 & x < L/2 \\ 1 & x > L/2 \end{cases}$.

   (b) $u(x,0) = 6 + 4\cos\dfrac{3\pi x}{L}$.

   (c) $u(x,0) = -3\cos\dfrac{8\pi x}{L}$.

5. In this problem, we apply the idea of separation of variables that we have learnt mainly within the context of the heat equation, to the wave equation. Consider the initial-boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = c^2\,\frac{\partial^2 u}{\partial x^2}$$

in the domain $0 < x < L$, $t > 0$, subject to the boundary and initial conditions

$$\begin{aligned} u(0,t) &= 0 \quad \text{for} \quad t \ge 0, \\ u(L,t) &= 0 \quad \text{for} \quad t \ge 0, \\ u(x,0) &= 0 \quad \text{for} \quad 0 \le x \le L, \\ \frac{\partial u}{\partial t}(x,0) &= 3\,c\,\sin\frac{4\pi x}{L} \quad \text{for} \quad 0 \le x \le L. \end{aligned}$$

   Here, the unknown is $u = u(x,t)$, and $c$ and $L$ are positive constants. This models a string tied at the endpoints $x = 0$ and $x = L$, and given an initial vertical velocity $3c\sin\frac{4\pi x}{L}$ at time zero when it is in a purely horizontal position.
   Solve the given initial-boundary value problem using separation of variables.