MATH2065: INTRO TO PDEs

Summer School 2012

Tutorial Solutions 6

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1. We have the equation
$$\phi''(x) + \lambda\,\phi(x) = 0\,,$$
which is a second-order constant-coefficient, homogeneous ODE. While this is *exactly* the equation we have looked at in class, the boundary conditions are different here, and therefore it is necessary to go through the whole argument to figure out possible values of λ leading to non-trivial solutions. The basic approach to follow is to attempt a solution of the form $\phi(x) = e^{\mu x}$ (note: we cannot use a λ here, since there is already one in the problem). This gives us the characteristic equation
$$\mu^2 + \lambda = 0\,.$$
Clearly, this will have different types of solutions for μ depending on whether λ is negative, zero, or positive, and so three different cases will need to be addressed.
<u>Case (i): $\lambda < 0$</u>
Set $\lambda = -q^2$, where $q > 0$. (This is just a convenient way of avoiding writing square-roots.) Thus, $\mu^2 - q^2 = 0$, and $\mu = \pm q$, leading to
$$\phi(x) = A\cosh qx + B\sinh qx\,.$$
(Using this form rather than the two exponentials e^{qx} and e^{-qx} turns out to be more convenient here.) Since $\phi(0) = 0$, $A = 0$. Since $\phi'(L) = 0$,
$$B\,q\,\cosh qL = 0\,.$$
Now, the cosh function is never zero, and hence it is not possible to find non-trivial solutions. That is, the only possibility here is $B = 0$, which means that $\phi(x)$ is identically the zero function. Therefore, there are no negative eigenvalues.
<u>Case (ii): $\lambda = 0$</u>
Then the ODE is simply $\phi''(x) = 0$, which can be integrated twice to give
$$\phi(x) = Ax + B\,.$$
Since $\phi(0) = 0$, $B = 0$. Since $\phi'(L) = 0$, $A = 0$. Thus, this yields only the trivial solution, and therefore $\lambda = 0$ is not an eigenvalue.
<u>Case (iii): $\lambda > 0$</u>
Set $\lambda = p^2$, where $p > 0$. Thus, $\mu^2 + p^2 = 0$, and so $\mu = \pm i\,p$. This gives the solutions $\cos px$ and $\sin px$, and hence the general solution
$$\phi(x) = A\cos px + B\sin px\,.$$
Now, we need $\phi(0) = 0$, and therefore $A = 0$. We then need $\phi'(L) = 0$, which means that
$$B\,p\,\cos px\Big|_{x=L} = 0 \qquad \Rightarrow \qquad \cos pL = 0\,.$$
Now, the cosine function is zero at $\pi/2$, and at any points $\pm\pi$, $\pm 2\pi$, $\pm 3\pi$, etc, away from this value. That is, we need
$$p\,L = \frac{\pi}{2} + n\,\pi \qquad n = 0, \pm 1, \pm 2, \pm 3, \cdots$$

However, we have chosen $p > 0$, and so we must limit ourselves to

$$p\,L = \frac{\pi}{2} + n\,\pi \qquad n = 0, 1, 2, 3, \cdots$$

This means that

$$\lambda_n = p^2 = \left(\frac{\pi/2 + n\pi}{L}\right)^2 \qquad n = 0, 1, 2, 3, \cdots$$

are the eigenvalues. The corresponding eigenfunctions are, since $A = 0$ but $B \neq 0$,

$$\phi_n(x) = \sin\left(\frac{\pi/2 + n\pi}{L} x\right) \qquad n = 0, 1, 2, \cdots$$

2. We have already solved the heat equation in this form, for a general initial condition $u(x, 0) = f(x)$. The solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} B_n \, \sin\frac{n\pi x}{L}\, e^{-k(n\pi/L)^2 t}\,,$$

where

$$B_n = \frac{2}{L}\int_0^L f(x)\, \sin\frac{n\pi x}{L}\, dx \tag{1}$$

gives the coefficients, which are related to the expansion of $f(x)$ in terms of sine functions in the form

$$f(x) = \sum_{n=1}^{\infty} B_n \, \sin\frac{n\pi x}{L}\,.$$

For this problem, we are given that

$$f(x) = 3\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}$$

Note that we do not need to invoke the formula (1), since f is already given in terms of a expansion in sines. In fact, we can simply read off

$$B_1 = 3 \qquad \text{and} \qquad B_3 = -1\,,$$

with all other B_ns being zero. This is since, for example, B_3 is the coefficient of $\sin\frac{3\pi x}{L}$. Therefore, in the infinite summation representing the solution $u(x, t)$, all but two terms are zero. Retaining only these two terms (corresponding to $n = 1$ and $n = 3$), we get the solution

$$u(x,t) = 3\sin\frac{\pi x}{L}\exp\left(-\frac{k\pi^2 t}{L^2}\right) - \sin\frac{3\pi x}{L}\exp\left(-\frac{9k\pi^2 t}{L^2}\right)\,.$$

3. Please refer the solution to the previous problem. In this case, however, we are not able to simply read off the values of the B_ns, since $f(x)$ is not given as an expansion over the sines. We therefore have to employ the formula (1):

$$\begin{aligned} B_n &= \frac{2}{L}\left(\int_0^{L/2} 1\,\sin\frac{n\pi x}{L}\,dx + \int_{L/2}^{L} 2\sin\frac{n\pi x}{L}\,dx\right) \\ &= \frac{2}{L}\left(-\frac{\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\Bigg|_{x=0}^{L/2} - 2\,\frac{\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\Bigg|_{x=L/2}^{L}\right) \\ &= -\frac{2}{L}\,\frac{L}{n\,\pi}\left(\cos\frac{n\pi}{2} - \cos 0 + 2\,\cos n\pi - 2\cos\frac{n\pi}{2}\right) \\ &= \frac{2}{n\,\pi}\left[1 - 2(-1)^n + \cos\frac{n\pi}{2}\right]\,. \end{aligned}$$

There is unfortunately no nice expression for $\cos\frac{n\pi}{2}$. The solution is therefore

$$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-2(-1)^n+\cos\frac{n\pi}{2}}{n}\sin\frac{n\pi x}{L}\,e^{-k(n\pi/L)^2 t}\,.$$

4. *(Very brief solutions will be given; the basic solution technique is essentially the same as in the previous problems, with the insulated boundary conditions requiring a cosine expansion rather than a sine one)*
The solution is given by (2.4.19), where the coefficients satisfy (2.4.21), and hence (2.4.23-24). The values of these coefficients is given below.

   (a) $A_0 = \frac{1}{L}\int_{L/2}^{L} 1\,dx = \frac{1}{2}$, $A_n = \frac{2}{L}\int_{L/2}^{L}\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_{L/2}^{L} = -\frac{2}{n\pi}\sin\frac{n\pi}{2}$ for $n \neq 0$.

   (b) By inspection $A_0 = 6$, $A_3 = 4$, and all other As are zero.

   (c) By inspection, $A_8 = -3$, and all other As are zero.

5. Assume a separation of variables solution of the form $u(x,t) = X(x)\,T(t)$, which upon substitution into the PDE leads to

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)} = -\lambda$$

for a separation constant λ. The fact that λ must be a constant is obtained by realising that it cannot be a function of t (since it equals the left-hand term above), and moreover it cannot be a function of x (since it equals the middle term above). Now, examining the *homogeneous* boundary and initial conditions, we see that

$$\begin{array}{rclcl} u(0,t)=0 & \Rightarrow & X(0)\,T(t)=0 \text{ for all } t & \Rightarrow & X(0)=0 \\ u(L,t)=0 & \Rightarrow & X(L)\,T(t)=0 \text{ for all } t & \Rightarrow & X(L)=0 \\ u(x,0)=0 & \Rightarrow & X(x)\,T(0)=0 \text{ for all } t & \Rightarrow & T(0)=0. \end{array}$$

This leads to the ODEs

$$\begin{aligned} X''(x) &= -\lambda\,X(x) \quad ; \quad X(0)=X(L)=0 \\ T''(t) &= -c^2\lambda\,T(t) \quad ; \quad T(0)=0. \end{aligned} \tag{2}$$

The ODE for $X(x)$ is the standard boundary value problem, which can be nontrivially solved only if $\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2$ for positive integers n, leading to the solutions

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right).$$

The T equation then becomes

$$T''(t) + \left(\frac{n\pi c}{L}\right)^2 T(t) = 0$$

from which the basic form

$$T_n(t) = A\cos\frac{n\pi ct}{L} + B\sin\frac{n\pi ct}{L}$$

results. Since $T(0) = 0$ is required, A must be zero, and the cosine term can be discarded. Now, since the solution $u(x,t)$ is the product of X and T, for each n the function

$$u_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right)$$

is a solution. The general solution is the superposition of all these:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right) .$$

The inhomogeneous initial condition is all that remains to be satisfied. Taking the t derivative of the above,

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} B_n \left(\frac{n\pi c}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right) .$$

This, when evaluated at $t = 0$, must be $3c\sin\frac{4\pi x}{L}$, and so

$$3c\sin\frac{4\pi x}{L} = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L}\sin\left(\frac{n\pi x}{L}\right).$$

It should be clear that the condition $3c = B_4\frac{4\pi c}{L}$ for B_4 is all that is necessary (all other $B's$ are zero). Thus, $B_4 = \frac{3L}{4\pi}$, and $B_n = 0$ for all other n. Thus, only one term (that corresponding to $n = 4$) survives from the superposed solution, and hence

$$u(x,t) = \frac{3L}{4\pi}\sin\frac{4\pi x}{L}\sin\frac{4\pi ct}{L}.$$