MATH2065: INTRO TO PDEs

Summer School 2012

Tutorial Solutions 7

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1. A product solution $\phi(x,y) = X(x)\,Y(y)$ is assumed, and substitution into the PDE yields

$$X''(x)Y(y) + X(x)Y''(y) = 0 \quad \Rightarrow \quad \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda,$$

where λ is the separation constant. (Note: an "aside" appears later on as to what would happen if you choose $(-\lambda)$ above instead of λ. It actually does not matter in the end.) Moreover, the homogeneous boundary conditions tell us that

$$\begin{array}{lllll} \phi(x,0) = 0 & \Rightarrow & X(x)\,Y(0) = 0 \text{ for all } x & \Rightarrow & Y(0) = 0 \\ \phi(x,L) = 0 & \Rightarrow & X(x)\,Y(L) = 0 \text{ for all } x & \Rightarrow & Y(L) = 0 \\ \phi(0,y) = 0 & \Rightarrow & X(0)\,Y(y) = 0 \text{ for all } y & \Rightarrow & X(0) = 0. \end{array}$$

Notice that we do not deal with the inhomogeneous condition yet; that will not give us straightforward conditions as above. Summarising the ODEs and the conditions which we have at this point:

$$\begin{array}{rcll} X''(x) & = & \lambda\,X(x) & ; \quad X(0) = 0, \\ Y''(y) & = & -\lambda\,Y(y) & ; \quad Y(0) = Y(L) = 0. \end{array}$$

We deal with the <u>second</u> equation first; the reason is that we have <u>two</u> conditions on this, but only one for the X equation (we will not be able to figure out conditions on λ with only one condition present). Now, examining the eigenvalue problem associated with the Y equation, we note that, as usual

$$\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

with corresponding eigenfunctions

$$Y_n(y) = \sin\frac{n\pi y}{L}.$$

With this choice of λ, the X equation becomes

$$X''(x) - \left(\frac{n\pi}{L}\right)^2 X(x) = 0,$$

for which the general solution is

$$X_n(x) = A_n \cosh\frac{n\pi x}{L} + B_n \sinh\frac{n\pi x}{L}.$$

(Equivalently, one may use the form $C_n \exp\left(\frac{n\pi x}{L}\right) + D_n \exp\left(-\frac{n\pi x}{L}\right)$ – but the above form turns out to be more convenient in this case.) Since $X(0) = 0$ is required, A_n must be zero. The linear superposition of the resulting product solutions $\phi_n(x,y) = X_n(x)\,Y_n(y)$ is

$$\phi(x,y) = \sum_{n=1}^{\infty} B_n \sinh\frac{n\pi x}{L} \sin\frac{n\pi y}{L}, \tag{1}$$

for yet to be determined constants B_n. The final boundary condition we need to satisfy is $\phi(L,y) = f(y)$, which when applied to Eq. (1) yields

$$\sum_{n=1}^{\infty} B_n \sinh{(n\pi)} \sin\frac{n\pi y}{L} = f(y).$$

This is the standard situation where a sum of $\sin(n\pi y/L)$ appears, except that here the coefficient is not simply B_n but $B_n \sinh{(n\pi)}$. Thus, the final form of the solution $u(x,t)$ will also have coefficients of this more complicated form instead of simply B_n. Thus we find

$$B_n \sinh{(n\pi)} = \frac{2}{L}\int_0^L f(y)\sin\frac{n\pi y}{L}dy\,.$$

Therefore, our unknown coefficients are given by

$$B_n = \frac{2}{L\sinh\,(n\pi)}\int_0^L f(y)\sin\frac{n\pi y}{L}dy. \tag{2}$$

The full solution is Eq. (1) with the B_n's given by Eq. (2).

Aside: You might have chosen $-\lambda$ as your initial separation constant. Had you done so, your equations for X and Y would be similar to those above, with λ replaced with $-\lambda$. Still, the Y equation would be the one to examine first, since it involves two boundary values. This equation would then be

$$Y''(y) = \lambda\, Y(y) \quad ; \quad Y(0) = Y(L) = 0\,.$$

This then would be not quite in the standard boundary-value representation we've used – the difference is that there would be a λ where we would normally see a $-\lambda$ on the right-hand side. If in the standard form, our eigenvalues would be $\left(\frac{n\pi}{L}\right)^2$. Since this has a negative sign difference, the eigenvalues then would be *negative* this value, i.e., $-\left(\frac{n\pi}{L}\right)^2$. The eigenfunctions would be the same. Now, the X equation, which is now

$$X''(x) = -\lambda\, X(x) \quad ; \quad X(0) = 0$$

with these new eigenvalues substituted in, becomes

$$X''(x) = \left(\frac{n\pi}{L}\right)^2 X(x)\,.$$

This is exactly the same equation that you would have got had you chosen the original separation constant as λ rather than $-\lambda$. Therefore, the final solution would be the same independently of whether you'd put in a $+\lambda$ or $-\lambda$ as your initial separation constant. The point is: it does not matter – you just need to be consistent with whatever choice you make.

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2. Try separation of variables in the form $u(x,y) = h(x)\phi(y)$. Looking at the zero boundary conditions, we see that

$$\begin{aligned}
\frac{\partial u}{\partial x}(0,y) &= 0 \text{ for all } y &\Rightarrow\quad& h'(0)\phi(y) = 0 \text{ for all } y &\Rightarrow\quad& h'(0) = 0\,.\\
\frac{\partial u}{\partial x}(L,y) &= 0 \text{ for all } y &\Rightarrow\quad& h'(L)\phi(y) = 0 \text{ for all } y &\Rightarrow\quad& h'(L) = 0\,.\\
u(x,0) &= 0 \text{ for all } x &\Rightarrow\quad& h(x)\phi(0) = 0 \text{ for all } x &\Rightarrow\quad& \phi(0) = 0\,.
\end{aligned}$$

We leave the nonhomogeneous boundary condition as it is for the moment, since we are not able to extract information on the functions h and ϕ from it. Now, substituting the "separation of variables" solution to the PDE,

$$\frac{1}{h}h''(x) = -\frac{1}{\phi}\phi''(y) = -\lambda\,.$$

The h equation is then in a familiar form:

$$\frac{d^2h}{dx^2} = -\lambda\, h(x) \quad ; \quad h'(0) = 0 \ \ , h'(L) = 0 \, .$$

Thus, from our table on boundary value problems, $\lambda_n = (n\pi/L)^2$, with $n = 0, 1, 2, \cdots$, and $h_n(x) = \cos\frac{n\pi x}{L}$. The ϕ equation becomes

$$\frac{d^2\phi}{dy^2} = \left(\frac{n\pi}{L}\right)^2 \phi(y)\, .$$

When $n = 0$, this has solution $\phi_0(y) = Ay + B$, and applying $\phi(0) = 0$ yields $B = 0$. When $n \neq 0$,

$$\phi(y) = C\cosh\frac{n\pi y}{L} + D\sinh\frac{n\pi y}{L}$$

(this is a more convenient representation than the exponentials), and applying $\phi(0) = 0$, we get $C = 0$. The result of superposition on all of these solutions is therefore

$$u(x, y) = A_0\, y + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\sinh\frac{n\pi y}{L}\, . \tag{3}$$

The nonhomogeneous boundary condition yields

$$f(x) = A_0 H + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\sinh\frac{n\pi H}{L}\, ,$$

which we identify as a series representation for f. We can now read off the coefficients based on our table on boundary value problems as being

$$\begin{aligned} A_0 H &= \frac{1}{L}\int_0^L f(x)dx \quad \Rightarrow \quad A_0 = \frac{1}{HL}\int_0^L f(x)dx\, , \\ A_n \sinh\frac{n\pi H}{L} &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \quad \Rightarrow \quad A_n = \frac{2}{L\sinh\frac{n\pi H}{L}}\int_0^L f(x)\cos\frac{n\pi x}{L}dx\, . \end{aligned}$$

The solution is therefore (3), with the constants as given above.

3. It helps to follow the example from the lecture in this case. The example was for *inside* a circle, whereas we now have to solve the equation *outside* a circle. When inside the circle, we got rid of the terms of the form r^{-n}, since these blew up (went to infinity) at the origin, which was inside the domain of interest. Now, when considering a domain outside the circle, we have no excuse to throw out these terms. On the other hand, we are given the hint that $u(r, \theta)$ should remain finite as $r \to \infty$. For this to happen, we certainly cannot tolerate having any terms of the form r^n. Therefore, we need to keep terms of the form r^{-n}, and chuck out those of the form r^n.
   There are still a couple of terms to worry about – those resulting from choosing $n = 0$. These give r solutions in the form $A_0 + B_0 \ln r$. Since we need the solution to be finite as $r \to \infty$, this logarithmic part is also not acceptable (although the constant A_0 is fine). Therefore, getting rid of all these unwelcome terms, we are left with

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} A_n r^{-n}\cos n\theta + \sum_{n=1}^{\infty} B_n r^{-n}\sin n\theta\, , \tag{4}$$

   for a bunch of (to be determined) constants.

(a) For this choice, substituting $r = a$ in (4),

$$\ln 2 + 4\cos 3\theta = A_0 + \sum_{n=1}^{\infty} A_n a^{-n} \cos n\theta + \sum_{n=1}^{\infty} B_n a^{-n} \sin n\theta\,.$$

There needs to be equality of the functions above, and by equating coefficients of the constant term, we get $A_0 = \ln 2$. Similarly equating coefficients of $\cos 3\theta$, we get $A_3 = 4a^3$. That takes care of everything, and we set all other constants equal to zero. The solution is then, from (4),

$$u(r,\theta) = \ln 2 + 4\left(\frac{a}{r}\right)^3 \cos 3\theta\,.$$

(b) *Cheap trick:* The boundary condition are the same as in the case inside the disk except that a^n is replaced by a^{-n}. Thus, the coefficients are determined from the solution of that case with a^n replaced by a^{-n}.