MATH2065: INTRO TO PDEs

Summer School 2012

Tutorial 8

There is no new material in this tutorial; Instead, you are being given a chance to revise previous work.

1. Consider the initial value problem

$$\frac{d\phi}{dt} - 5\,\phi = \sin t + (t+1)\,e^{5t} \qquad ; \qquad \phi(0) = 0\,,$$

where the unknown is $\phi(t)$. In First Year maths, you learnt how to solve first-order ODEs of this form using an *integrating factor*. In this question, you are asked to solve the above initial value problem using techniques that you have learnt in MATH2065. That is, determine the corresponding *homogeneous solution* $\phi_h(t)$ and the *particular solution* $\phi_p(t)$ (using undetermined coefficients), and thereby determine the solution $\phi(t)$ to the initial value problem. (Note that although we have studied these methods in MATH2065 within the context of *second-order* ODEs, the ideas are clearly applicable – and in fact somewhat easier to apply – to *first-order* ODEs).

2. Use Laplace transforms to solve the following differential equation for $y(t)$:

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} - 6y = \begin{cases} e^{-t}, & 0 < t < 3, \\ 0, & t > 3, \end{cases}$$

given that $y(0) = 0$ and $y'(0) = 2$.

3. Solve the initial-boundary value problem:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, && \text{for} \quad 0 < x < \pi \quad , \quad t > 0 \\ u(0,t) &= 0, && \text{(left boundary condition)} \\ \frac{\partial u}{\partial x}(\pi,t) &= 0, && \text{(right boundary condition)} \\ u(x,0) &= f(x), && \text{(initial condition)} \end{aligned}$$

This models the temperature variation in a rod of length π, initially at a temperature given by $f(x)$, then positioned with one end in ice and the other end insulated.

(a) Determine the steady-state solution, $\bar{u}(x)$, to the problem.

(b) Using the method of separation of variables, find the general solution $u(x,t)$ to the PDE plus boundary conditions as an infinite series involving unknown coefficients A_n.

(c) Find an expression for A_n in terms of the initial condition function $f(x)$. You may assume the result

$$\int_0^\pi \sin\left(\frac{2n+1}{2}x\right)\,\sin\left(\frac{2m+1}{2}x\right)\,dx = \begin{cases} 0, & n \neq m, \\ \pi/2, & n = m, \end{cases}$$

for n, m integers.

(d) Find $u(x,t)$ for the case where $f(x) = 100$.