MATH2065: INTRO TO PDEs

Summer School 2012

Tutorial Solutions 8

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1. The homogeneous solution satisfies

$$\frac{d\phi_h}{dt} - 5\phi_h = 0.$$

The auxiliary equation is $\lambda - 5 = 0$, so $\lambda = 5$. Therefore

$$\phi_h(t) = Ce^{5t}$$

where C is an arbitrary constant.

Let $\phi_{p1}(t)$ satisfy

$$\frac{d\phi_{p1}}{dt} - 5\phi_{p1} = \sin t.$$

For a solution, try

$$\phi_{p1}(t) = A\sin t + B\cos t.$$

Therefore

$$\phi'_{p1}(t) = A\cos t - B\sin t.$$

Substituting,

$$A\cos t - B\sin t - 5A\sin t - 5B\cos t = \sin t.$$

Equating coefficients of $\cos t$ and $\sin t$ gives $A - 5B = 0$ and $-B - 5A = 1$ with solution $A = -5/26$, $B = -1/26$. Therefore

$$\phi_{p1}(t) = -\frac{5}{26}\sin t - \frac{1}{26}\cos t.$$

Let $\phi_{p2}(t)$ satisfy

$$\frac{d\phi_{p2}}{dt} - 5\phi_{p2} = (t+1)e^{5t}.$$

For a solution, try (note that e^{5t} is contained in $\phi_h(t)$)

$$\phi_{p2}(t) = (Dt^2 + Et)e^{5t}.$$

Then

$$\phi'_{p2}(t) = (2Dt + E + 5Dt^2 + 5Et)e^{5t}.$$

Substituting:

$$2Dt + E + 5Dt^2 + 5Et - 5Dt^2 - 5Et = t + 1$$

leading to (note the cancellation of the t^2 term) $D = 1/2$, $E = 1$. Therefore

$$\phi_{p2}(t) = (\frac{1}{2}t^2 + t)e^{5t}.$$

The general solution is therefore

$$\phi(t) = Ce^{5t} - \frac{5}{26}\sin t - \frac{1}{26}\cos t + (\frac{1}{2}t^2 + t)e^{5t}$$

and $\phi(0) = 0$ gives $C = 1/26$.

2. The transformed equation is

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + 5[sY(s) - y(0)] - 6Y(s) &= \int_0^3 e^{-st-t}dt \\ &= \frac{1}{s+1}\left(1 - e^{-3(s+1)}\right). \end{aligned}$$

Therefore

$$(s^2 + 5s - 6)Y(s) - 2 = \frac{1}{s+1}\left(1 - e^{-3(s+1)}\right).$$

Solving for $Y(s)$, and using $s^2 + 5s - 6 = (s-1)(s+6)$ leads to

$$\begin{aligned} Y(s) &= \frac{3s+2}{(s+1)(s-1)(s+6)} - \frac{1}{(s+1)(s-1)(s+6)}e^{-3(s+1)} \\ &= \frac{-\frac{1}{10}}{s+1} + \frac{\frac{5}{14}}{s-1} + \frac{-\frac{9}{35}}{s+6} - \left(\frac{-\frac{1}{10}}{s+1} + \frac{\frac{1}{14}}{s-1} + \frac{\frac{1}{35}}{s+6}\right)e^{-3s}e^{-3}. \end{aligned}$$

Inverting the transform gives

$$y(t) = -\frac{1}{10}e^{-t} + \frac{5}{14}e^{t} - \frac{9}{35}e^{-6t} - e^{-3}H(t-3)\left(-\frac{1}{10}e^{-(t-3)} + \frac{1}{14}e^{t-3} + \frac{1}{35}e^{-6(t-3)}\right)$$

where the second term follows from the "t-shifting" formula.

3. (a) For the steady-state, $\partial\bar{u}/\partial t = 0$, so the heat equation reduces to $d^2\bar{u}/dx^2 = 0$ which has solution $\bar{u}(x) = Ax + B$. The condition $\bar{u}(0) = 0$ gives $B = 0$. Therefore $\bar{u}'(x) = A$ and the condition $\bar{u}'(\pi) = 0$ gives $A = 0$. Therefore $\bar{u}(x) = 0$. (Note that this also follows on physical grounds - all the heat eventually drains from the $x = 0$ end of the bar.)

   (b) Let $u(x,t) = X(x)T(t)$. Then $XT' = X''T$ and dividing by XT gives

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda.$$

   We thus have the two ordinary differential equations $X'' + \lambda X = 0$ and $T' = -\lambda T$. Since there are zero boundary conditions at each end it is clear that a sinusoidal solution is needed, so choose $\lambda > 0$. Thus the solutions of these equations are

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x) \quad \text{and} \quad T(t) = Ce^{-\lambda t}.$$

   $u(0,t) = 0$ gives $X(0) = 0$ so $A = 0$. Therefore $X'(x) = B\sqrt{\lambda}\cos(\sqrt{\lambda}x)$ and the condition $\partial u(\pi,t)/\partial x = 0$ gives $X'(\pi) = 0$ and hence

$$B\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) = 0.$$

   Choosing $B = 0$ just gives the trivial solution $X(x) = 0$, so we must have $\cos(\sqrt{\lambda}\pi) = 0$, which gives

$$\sqrt{\lambda}\pi = (2n+1)\frac{\pi}{2}, \quad n = 0, 1, 2, \ldots.$$

   A separable solution is thus

$$u_n(x,t) = A_n \sin\left(\frac{2n+1}{2}x\right)\exp\left(-\left(\frac{2n+1}{2}\right)^2 t\right), \quad n = 0, 1, 2, \ldots$$

   and the general solution is

$$u(x,t) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{2n+1}{2}x\right)\exp\left(-\left(\frac{2n+1}{2}\right)^2 t\right).$$

(c) The initial condition $u(x,0) = f(x)$ gives

$$f(x) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{2n+1}{2}x\right).$$

Multiplying each side of this equation by $\sin((2m+1)x/2)$ and integrating from 0 to ∞ with respect to x gives

$$\begin{aligned}
\int_0^{\infty} f(x)\sin\left(\frac{2m+1}{2}x\right)dx &= \sum_{n=0}^{\infty} A_n \int_0^{\pi} \sin\left(\frac{2n+1}{2}x\right)\sin\left(\frac{2m+1}{2}x\right)dx \\
&= A_m\frac{\pi}{2}, \quad \text{since all other terms are zero.}
\end{aligned}$$

Therefore

$$A_n = \frac{2}{\pi}\int_0^{\infty} f(x)\sin\left(\frac{2n+1}{2}x\right)dx$$

(d)

$$\begin{aligned}
A_n &= \frac{2}{\pi}100\int_0^{\infty}\sin\left(\frac{2n+1}{2}x\right)dx \\
&= \frac{200}{\pi}\frac{2}{2n+1}\left(-\cos\left(\frac{2n+1}{2}\pi\right)+\cos(0)\right) \\
&= \frac{400}{(2n+1)\pi}, \quad \text{since} \quad \cos\left(\frac{2n+1}{2}\pi\right) = 0,
\end{aligned}$$

Thus

$$u(x,t) = \frac{400}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\left(\frac{2n+1}{2}x\right)\exp\left(-\left(\frac{2n+1}{2}\right)^2 t\right)$$