MATH2065: INTRO TO PDEs

Summer School 2012

Tutorial Questions 9

Questions marked with the dagger symbol † are intentionally more challenging.

1. For the following functions, sketch the Fourier series of $f(x)$ on the interval $-L \leq x \leq L$ and determine the Fourier coefficients:
(a) $f(x) = x$
(b) $f(x) = \sin(\pi x/L)$
(c) $f(x) = H(x)$, the Heaviside function, which is 0 for $x < 0$ and 1 for $x > 0$.

2. (a) Let $f(x) = \cos x$ on the interval $(0, \pi)$. Sketch the odd periodic extension of $f(x)$ in the range $-3\pi < x < 3\pi$.
   (b) Determine the Fourier sine series $f_s(x)$ for the function $f(x) = \cos x$ on the interval $(0, \pi)$.
   (c) Is the graph of $f_s(x)$ identical to the graph you presented in part (a)? Why does the Fourier series that you have found for the *cosine* function involve only *sine* terms?

3. Suppose a $2L$-periodic function possesses a Fourier series
$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right] .$$
   (a) By taking the inner product of the above expression with $f(x)$, and also utilising the formulæ for the Fourier coefficients, derive *Parseval's identity*
$$\frac{1}{L} \int_{-L}^{L} [f(x)]^2 \, dx = 2a_0^2 + \sum_{n=1}^{\infty} \left[a_n^2 + b_n^2 \right] .$$
   (b) † Does the fact that there is a $\sim$ sign rather than an equality between the function $f(x)$ and its Fourier series interfere with your derivation in part (a)?
   (c) The function $g(x) = x^2$ on $(-1, 1)$ is periodically extended to form a 2-periodic function. You are *given* that its Fourier series is
$$g(x) \sim \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi x) .$$
   Applying Parseval's identity to g, determine the value of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$.

4. † Let
$$f(x) = \cos x \,, \quad \text{on} \quad [0, \alpha] \,,$$
where α is some constant, $0 < \alpha < \pi$.
   (a) Draw the graph of the even periodic extension of $f(x)$ with period 2α, and then compute the Fourier cosine series associated with f.
   (b) What happens when $\alpha \to \pi$?