MATH2065: INTRO TO PDEs

Summer School 2012

Tutorial Solutions 9

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1. Here, we will just focus on determining the Fourier coefficients.

   (a) Within the interval $(-L, L)$, we have the Fourier series

$$x \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \, .$$

   $a_0 = 0$ since $f(x)$ is odd. Similarly, $a_n = 0$ for $n \neq 0$, again since the integrand in the formula for a_n is odd. It remains to find b_n, which is given by

$$b_n = \frac{1}{L} \int_{-L}^{L} x \sin \frac{n\pi x}{L} \, dx = \frac{2}{L} \int_{0}^{L} x \sin \frac{n\pi x}{L} \, dx \, ,$$

   since the integrand is even. Integrating by parts,

$$b_n = \frac{2}{L} \left[\frac{x \cos \frac{n\pi x}{L}}{-\frac{n\pi}{L}} \bigg|_0^L - \int_0^L \frac{\cos \frac{n\pi x}{L}}{-\frac{n\pi}{L}} \, dx \right]$$

   which eventually yields (after using the fact that $\cos n\pi = (-1)^n$),

$$b_n = \frac{2L}{n\pi} (-1)^{n+1} \, .$$

   Therefore, the Fourier series is

$$x \sim \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{L} \, .$$

   (b) This is a trick question. The whole purpose of obtaining a Fourier series is to represent a function in terms of sines and cosines. This function is already in the correct form! Therefore, its Fourier series is simply $\sin \frac{\pi x}{L}$.
   Alternatively, one may have said that it is necessary to determine the values of the constants satisfying

$$\sin \frac{\pi x}{L} \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \, .$$

   Clearly, we can choose $b_1 = 1$, with all other constants (as and bs) being zero. (This is equivalent to equating coefficients in the above expression.)

   (c) We employ the formula for a_n and note that the function is zero in the interval $(-L, 0)$. Therefore

$$a_0 = \frac{1}{2L} \int_0^L dx = \frac{1}{2} \, .$$

   Moreover, for $n \neq 0$,

$$a_n = \frac{1}{L} \int_0^L \cos \frac{n\pi x}{L} dx = \frac{1}{L} \frac{L}{n\pi} \sin \frac{n\pi x}{L} \bigg|_0^L = 0 \, ,$$

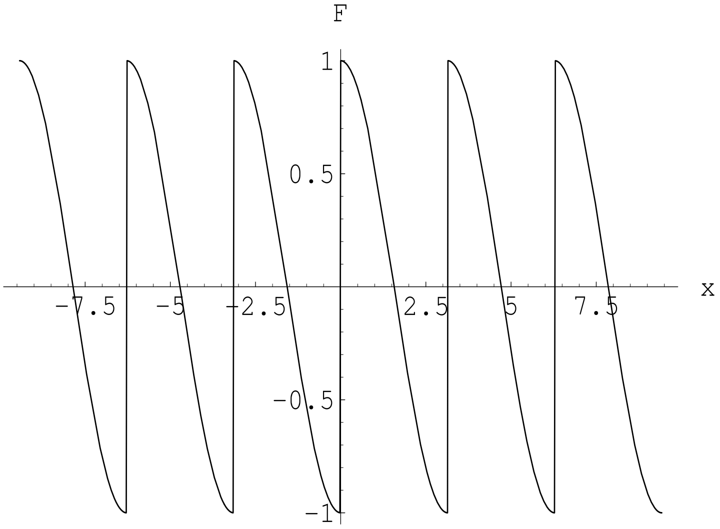
since $\sin n\pi = 0$ for integers n. Also

$$b_n = \frac{1}{L}\int_0^L \sin\frac{n\pi x}{L}dx = \frac{-1}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^L = \frac{1-(-1)^n}{n\pi}\,.$$

Thus, $b_n = 2/n\pi$ for odd n, but $b_n = 0$ for even n. The Fourier series is thus representible as

$$f(x) \sim \frac{1}{2} + \frac{2}{\pi}\sum_{n=1,\text{odd}}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{L}\,.$$

2. (a) The odd periodic extension of $f(x) = \cos x$ is given below. We first extend the function to $(-\pi, 0)$ by doing an odd extension (reflection about the origin, or alternatively, successive reflections about the x and y axes). Thereafter, we take this function defined on an interval of length 2π, and simply make copies of it.



(b) We use the formula for the Fourier sine coefficients with $L = \pi$, and the trigonometric identity $\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$.

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi \cos x \sin nx\,dx = \frac{1}{\pi}\int_0^\pi [\sin(n{+}1)x \;+\; \sin(n{-}1)x]\,dx \\ &= \frac{1}{\pi}\Big[-\frac{\cos(n{+}1)x}{n{+}1} - \frac{\cos(n{-}1)x}{n{-}1}\Big]_0^\pi \qquad (\text{if } n \neq 1) \\ &= \frac{1}{\pi}\Big[-\frac{(-1)^{n+1}-1}{n{+}1} - \frac{(-1)^{n-1}-1}{n{-}1}\Big] \\ &= \begin{cases} \dfrac{4}{\pi}\dfrac{n}{n^2-1} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}\,. \end{aligned}$$

We need to consider the case $n = 1$ separately, since we have apparently “divided by zero” above if $n = 1$. Evaluating it from the initial b_n equation,

$$b_1 = \frac{2}{\pi}\int_0^\pi \cos x \sin x\,dx = \frac{1}{\pi}\int_0^\pi \sin 2x\,dx = 0\,.$$

On replacing n by $2m$, the Fourier sine series can be rewritten as

$$f_s(x) = \sum_{n=1}^{\infty} b_n \sin nx = \frac{8}{\pi}\sum_{m=1}^{\infty}\frac{m\sin 2mx}{4m^2-1}\,.$$

(c) We note that the function above "almost" has the graph of the odd periodic extension shown in the above figure. The only difference is that at points of jump discontinuity (i.e., at $x = n\pi$ for integers n), the value of the Fourier sine series is the average of the left and right hand limits. In other words, $f_s(n\pi) = 0$ at all integer values n.
The series we computed is the Fourier *sine* series for the function $\cos x$ on $(0, \pi)$. Note that the function, although given by $\cos x$ in the interval $(0, \pi)$, when extended to $\mathbb{R}$ is an *odd* function. The fact that it is derived from a cosine function in the base interval is of no consequence. Since, as a function on $\mathbb{R}$, it is odd, it should contain only sine terms.

3. (a) Taking the "inner-product" means multiplying by the function, and then integrating from $-L$ to L (the interval of interest). Doing so, and simply treating the $\sim$ sign as an equality, we get

$$\int_{-L}^{L} [f(x)]^2\, dx = a_0 \int_{-L}^{L} f(x) dx + \sum_{n=1}^{\infty} \left[a_n \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx + b_n \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx \right]. \quad (1)$$

Now, recall our Fourier coefficient equations, which are

$$\begin{aligned} a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x)\, dx & \Rightarrow \quad & \int_{-L}^{L} f(x)\, dx = 2La_0\,, \\ a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx & \Rightarrow \quad & \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx = La_n\,, \quad (n \neq 0) \\ b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx & \Rightarrow \quad & \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx = Lb_n\,. \end{aligned}$$

Substituting for the integrals in (1), we get

$$\int_{-L}^{L} [f(x)]^2\, dx = 2La_0^2 + \sum_{n=1}^{\infty} \left[La_n^2 + Lb_n^2 \right].$$

Dividing the above expression by L gives us Parseval's identity.

(b) In the above part, we essentially treated the $\sim$ sign as an equality. Recall that the convergence theorem tells us that at all points of continuity of f, equality is in fact valid. The only case when equality is not valid is at points of jump discontinuity, when the Fourier series is equal to $[f(x_+) + f(x_-)]/2$. However, what we did in the previous section was take an integral over the interval $(-L, L)$. Such points of jump discontinuity do not add to the integral, since they make no contribution to the area under the curve. Therefore, we may as well ignore them.

(c) We see that $b_n = 0$, $a_0 = 1/3$ and

$$a_n = \frac{4\,(-1)^n}{n^2\,\pi^2} \qquad (n \neq 0)$$

for this function. Also, $L = 1$ for this problem. Substituting these values into Parseval's identity,

$$\frac{1}{1} \int_{-1}^{1} \left[x^2\right]^2\, dx = 2\left(\frac{1}{3}\right)^2 + \sum_{n=1}^{\infty} \left[\frac{4\,(-1)^n}{n^2\,\pi^2} \right]^2.$$

Noting that x^2 is an even function, and that $(-1)^{2n} = 1$ since the exponent is even, we can rewrite the above as

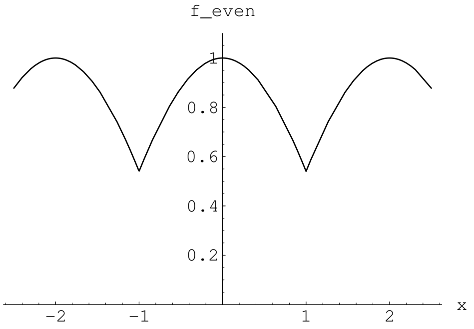
$$2\int_0^1 x^4\,dx = \frac{2}{9} + \frac{16}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}\,.$$

Therefore,

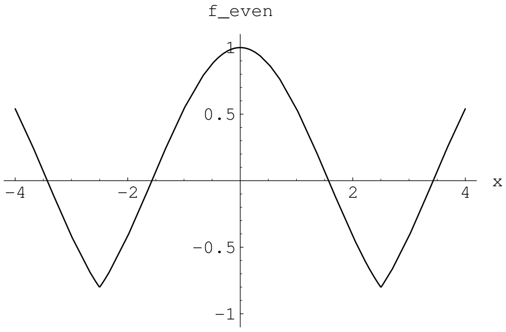
$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{16}\left[2\left(\frac{1}{5}\right) - \frac{2}{9}\right] = \frac{\pi^4}{90}\,.$$

4. (a) Consider the function $f(x) = \cos x$ on $[0, \alpha]$. Two examples of the even periodic extension, with period 2α, $0 < \alpha < \pi$, are shown below.

example 1: $\alpha = 1$



example 2: $\alpha = 2.5$



The Fourier coefficients are

$$a_0 = \frac{1}{\alpha}\int_0^{\alpha}\cos x\,dx = \frac{\sin\alpha}{\alpha},$$

and for $n = 1, 2, \cdots$,

$$\begin{aligned} a_n &= \frac{2}{\alpha}\int_0^{\alpha}\cos x\cos\frac{n\pi x}{\alpha}dx \\ &= \frac{1}{\alpha}\int_0^{\alpha}\left\{\cos\Bigl(\frac{n\pi}{\alpha}+1\Bigr)x + \cos\Bigl(\frac{n\pi}{\alpha}-1\Bigr)x\right\}dx = (-1)^{n-1}\,\frac{2\alpha\sin\alpha}{n^2\pi^2-\alpha^2}\,. \end{aligned}$$

The Fourier cosine series for this function is therefore:

$$\frac{\sin\alpha}{\alpha} + \sum_{n=1}^{\infty}(-1)^{n-1}\,\frac{2\alpha\sin\alpha}{n^2\pi^2-\alpha^2}\cos\frac{n\pi x}{\alpha}\,.$$

(b) As $\alpha \to \pi$, all the a_ns tend to zero except a_1. L'Hôpital's rule shows that a_1 tends to 1. Hence the Fourier series reduces to the single term, $\cos x$.