SS2065: INTRO TO PDEs
Brief Answers to Tutorial Questions

Tutorial 1

1. (a) $x = 2e^{3t}$.
   (b) $y = 10e^{5(1-z)}$.
   (c) $x = \frac{4}{3}(1 - e^{-3t})$

2. (a) $y(x) = C_1 \exp(7x)$
   (b) $y(x) = C_1 \exp(x) + C_2 \exp(-2x)$
   (c) $y(x) = C_1 \exp(0x) + C_2 \exp(3x) = C_1 + C_2 \exp(3x)$
   (d) $y(x) = C_1 \exp(2x) + C_2 \exp(-2x)$
   (e) $y(x) = C_1 \sin 2x + C_2 \cos 2x$
   (f) $y(x) = C_1 \exp(-3x) + C_2 x \exp(-3x)$
   (g) $y(x) = C_1 \exp(3x) \cos(4x) + C_2 \exp(3x) \sin(4x)$
   (h) $y(x) = C_1 \exp(x) + C_2 \exp(-x) + C_3 \exp(-2x)$

3. (a) $y(x) = \exp(7x)$
   (b) $y(x) = \frac{1}{3}[\exp(3x) - 1]$
   (c) $y(x) = x \exp(-3x)$
   (d) $y(x) = \exp(3x) \sin(4x)$

Tutorial 2

1. (a) $y_p = -\frac{1}{3}e^x$
   (b) $y_p(x) = -\frac{1}{9}x - 2$
   (c) $y_p(x) = -\cos x - \sin x$
   (d) $y_p(x) = x^2 - 2$
   (e) $y_p = \frac{1}{3}xe^{3x}$

2. (a) $y(x) = \frac{7}{5}e^{2x} - \frac{3}{20}e^{-3x} - \frac{5}{4}e^x$
   (b) $y(x) = -\frac{37}{36}e^{-2x} + \frac{26}{25}e^{-x} + (\frac{1}{30}x - \frac{11}{900})e^{4x}$
   (c) $y(x) = -\frac{2}{5}e^{3x} + e^x + \frac{1}{5}\sin x + \frac{2}{5}\cos x$

3. (a) $y_p = \frac{1}{3}x - \frac{10}{9} + \frac{1}{10}\sin x$
   (b) $y_p = \frac{1}{6}e^x + \frac{1}{10}\sin x - \frac{1}{10}\cos x$
   (c) $y_p = -\frac{1}{2}xe^x - e^{2x}$

4. (a) $x_h(t) = C_1 \cos(2t) + C_2 \sin(2t)$
   (b)

$$x(t) = \begin{cases} \dfrac{1}{4-\omega^2}\cos(\omega t), & (\omega \neq 2) \\ \dfrac{t}{4}\sin(2t), & (\omega = 2) \end{cases}$$

(c)

$$x(t) = \begin{cases} C_1 \cos(2t) + C_2 \sin(2t) + \dfrac{1}{4-\omega^2}\cos(\omega t) & (\text{if } \omega \neq 2) \\ C_1 \cos(2t) + C_2 \sin(2t) + \dfrac{t}{4}\sin(2t) & (\text{if } \omega = 2) \end{cases}$$

Tutorial 3

1. (a) $\dfrac{6}{(s+2)^4}$

   (b) $\dfrac{8s}{(s^2+16)^2}$

   (c) $\dfrac{e^{-3s}}{s}$

   (d) $\dfrac{4}{(s-3)^2+16}$

   (e) $\dfrac{(s+4)^2-36}{[(s+4)^2+36]^2}$

   (f) $e^{-5s}\left(\dfrac{2}{s^3}+\dfrac{10}{s^2}+\dfrac{25}{s}\right) - e^{-8s}\left(\dfrac{2}{s^3}+\dfrac{16}{s^2}+\dfrac{64}{s}\right)$

   (g) $\dfrac{24e^{-s}}{s^5}$

2. (a) $\frac{1}{2}\sin 2t$

   (b) $\frac{1}{2}H(t-3)\sin[2(t-3)]$

   (c) $\frac{1}{6!}t^6e^{4t}$

   (d) $-\frac{1}{6}e^{-t}+\frac{7}{6}e^{-7t}$

   (e) $\frac{1}{3}\cos t - \frac{1}{3}\cos 2t$

   (f) $\frac{2}{9}-\frac{2}{9}\cos 3t+\frac{1}{3}\sin 3t - 5H(t-4)\left(\frac{2}{9}-\frac{2}{9}\cos[3(t-4)]+\frac{1}{3}\sin[3(t-4)]\right)$

3.

4. $y(t) = 1 + e^{-t}$

5. $y(t) = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t$

6. $y(t) = g(t) - g(t-4)H(t-4)$ where $g(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$.

Tutorial 4

1. (a) $i(t) = \dfrac{V_0}{R}\left[(1-e^{R(t-a)/L})H(t-a) - (1-e^{R(t-b)/L})H(t-b)\right]$

   (b) $i(t) = \dfrac{V_0}{L}H(t-a)e^{-R(t-a)/L}$

2. $y(t) = 2e^{-t} - 2e^{-3t} - e^{-8}H(t-8)\left(2e^{-(t-8)} - 2e^{-3(t-8)}\right) + \dfrac{2}{3}H(t-8)\left(1-e^{-3(t-8)}\right) + e^{-3t}$

3.

4. $\dfrac{\sin t - t\,\cos t}{2}$

5. $y(t) = e^t + \displaystyle\int_0^t \sinh(t-\bar{t})\,g(\bar{t})\,d\bar{t}$

6. $F(s)/s$

7. (a) $b\cos 2b$ if $b \in (-1,3)$, zero otherwise.

Tutorial 5

1.

2.

3. $u(x) = -\frac{T}{L}\,x + T$

4. $u(x) = -\dfrac{x^4}{12} + \dfrac{L^3\,x}{3} + T$

5.

6. (a) $\dfrac{dh}{dt} = -\lambda\,k\,h$ and $\dfrac{1}{r}\dfrac{d}{dr}\left(r\,\dfrac{d\phi}{dr}\right) = -\lambda\,\phi$

(b) $\phi''(x) = -\lambda\,\phi(x)$ and $h''(y) = \lambda\,h(y)$

(c) $h'(t) = \lambda\,k\,h(t)$ and $\phi''''(x) = \lambda\,\phi(x)$

Tutorial 6

1.

$$\lambda_n = p^2 = \left(\frac{\pi/2 + n\pi}{L}\right)^2 \qquad n = 0,1,2,3,\cdots$$

$$\phi_n(x) = \sin\left(\frac{\pi/2 + n\pi}{L}x\right) \qquad n = 0,1,2,\cdots$$

2.

$$u(x,t) = 3\sin\frac{\pi x}{L}\exp\left(-\frac{k\pi^2 t}{L^2}\right) - \sin\frac{3\pi x}{L}\exp\left(-\frac{9k\pi^2 t}{L^2}\right).$$

3.

$$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1 - 2(-1)^n + \cos\frac{n\pi}{2}}{n}\,\sin\frac{n\pi x}{L}\,e^{-k(n\pi/L)^2 t}.$$

4. (a) $A_0 = \frac{1}{2}$, $A_n = -\frac{2}{n\pi}\sin\frac{n\pi}{2}$ for $n \neq 0$.

(b) $A_0 = 6$, $A_3 = 4$, all other As are zero.

(c) $A_8 = -3$, all other As are zero.

5.

$$u(x,t) = \frac{3L}{4\pi} \sin \frac{4\pi x}{L} \sin \frac{4\pi ct}{L}$$

Tutorial 7

1.

$$\phi(x,y) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi x}{L} \sin \frac{n\pi y}{L},$$

where the coefficients are given by

$$B_n = \frac{2}{L \sinh\, (n\pi)} \int_0^L f(y) \sin \frac{n\pi y}{L} dy\,.$$

2.

$$u(x,y) = A_0\, y + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \sinh \frac{n\pi y}{L}$$

where the coefficients are given by

$$A_0 = \frac{1}{HL} \int_0^L f(x) dx, \qquad A_n = \frac{2}{L \sinh \frac{n\pi H}{L}} \int_0^L f(x) \cos \frac{n\pi x}{L} dx\,.$$

3. (a)

$$u(r,\theta) = \ln 2 + 4 \left(\frac{a}{r}\right)^3 \cos 3\theta\,.$$

(b)

$$A_n = a^n a_n, \quad B_n = a^n b_n$$

where a_n and b_n are the coefficients of the Fourier series of $f(\theta)$.

Tutorial 8

1.

$$\phi(t) = \frac{1}{26} e^{5t} - \frac{5}{26} \sin t - \frac{1}{26} \cos t + (\frac{1}{2} t^2 + t) e^{5t}$$

2.

$$y(t) = -\frac{1}{10} e^{-t} + \frac{5}{14} e^{t} - \frac{9}{35} e^{-6t} - e^{-3} H(t-3) \left(-\frac{1}{10} e^{-(t-3)} + \frac{1}{14} e^{t-3} + \frac{1}{35} e^{-6(t-3)} \right)$$

3. (a) $\bar{u}(x) = 0$

(b)

$$u(x,t) = \sum_{n=0}^{\infty} A_n \sin \left(\frac{2n+1}{2} x \right) \exp \left(- \left(\frac{2n+1}{2} \right)^2 t \right)$$

(c)

$$A_n = \frac{2}{\pi}\int_0^\infty f(x)\sin\left(\frac{2n+1}{2}x\right)dx$$

(d)

$$u(x,t) = \frac{400}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\left(\frac{2n+1}{2}x\right)\exp\left(-\left(\frac{2n+1}{2}\right)^2 t\right)$$

Tutorial 9

1. The Fourier Series are:
   (a) $x \sim \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{L}$.
   (b) $\sin\frac{\pi x}{L}$
   (c) $f(x) \sim \frac{1}{2} + \frac{2}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{L}$.
2. (b) $f_s(x) \sim \frac{8}{\pi}\sum_{m=1}^{\infty}\frac{m\sin 2mx}{4m^2-1}$.
3. (c) $\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$
4. (a) $\frac{\sin\alpha}{\alpha} + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{2\alpha\sin\alpha}{n^2\pi^2-\alpha^2}\cos\frac{n\pi x}{\alpha}$
   (b) The Fourier series reduces to the single term, $\cos x$.

Tutorial 10

1. (d) $x \sim -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$
2. $c_n = \frac{1}{2in\pi\Delta}\exp\left(\frac{in\pi x_0}{L}\right)\left[\exp\left(\frac{in\pi\Delta}{L}\right) - 1\right]$

Tutorial 11

2. $\frac{2\alpha}{x^2+\alpha^2}$
4. $\frac{i}{\pi}\left[\frac{\sin\omega}{\omega^2} - \frac{\cos\omega}{\omega}\right]$
6. $\delta(\omega-\beta)$

Tutorial 12

1. $\frac{70}{\sqrt{1+2t}}\exp\left(-\frac{x^2}{2+4t}\right)$
2. $\int_{-\infty}^{\infty}F(\omega)\,e^{ik\omega^3 t}\,e^{-i\omega x}\,d\omega$
3. $\frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty}f(\bar{x})\,\exp\left[-\frac{(x-\bar{x}+ct)^2}{4kt}\right]\,d\bar{x}$
4. $\frac{1}{\sqrt{4\pi kt}}\exp\left[-\frac{(x+ct)^2}{4kt}\right]$
5. $\int_{-\infty}^{\infty}\frac{F(\omega)}{\sinh\,\omega}\sinh\,(\omega y)\,e^{-i\omega x}\,d\omega$

Tutorial 13

2. (d) $u(x,t) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\,\pi\,x}{L}\right) \sin\left(\frac{c\sqrt{n^2\pi^2-1}}{L}\,t\right)$

   (e) $\alpha_n = \frac{2}{c\sqrt{n^2\,\pi^2-1}} \int_0^L f(x)\,\sin\left(\frac{n\,\pi\,x}{L}\right) \mathrm{d}x$

3. $x(t) = \frac{3}{2}\,(\cosh t - \cos t)$, $\quad y(t) = \frac{3}{2}\sinh t + \frac{1}{2}\sin t$