1. Determine which of the series below converge, and which diverge.

   \( \sum_{n=1}^{\infty} \frac{1}{2n^2 + n + 1} \), \( \sum_{n=1}^{\infty} \frac{1}{2n^2 - n + 1} \), \( \sum_{n=1}^{\infty} \frac{1}{2n - 1} \),

   \( \sum_{n=1}^{\infty} \frac{1}{1 + 3\sqrt{n}} \), \( \sum_{n=0}^{\infty} \frac{2^n - 1}{3^n + 1} \), \( \sum_{n=1}^{\infty} \frac{2^n + 1}{3^n - 1} \),

   \( \sum_{n=2}^{\infty} \frac{1}{n^2 \log n} \), \( \sum_{n=1}^{\infty} \frac{\log n}{n^2} \).

2. For \( x \geq 0 \), and \( n \in \mathbb{N} \) let \( c_n = \frac{x^n}{1 + x^n} \).

   (i) Determine for which \( x \geq 0 \) the sequence \( (c_n) \) converges and find the limit as function of \( x \).

   (ii) Investigate how the convergence of the series \( \sum_{n=0}^{\infty} c_n \) depends on \( x \).

3. The recurrence \( x_{n+1} = \sqrt{2 + x_n}; \quad x_1 = \sqrt{2} \), generates the sequence

   \( x_n : \sqrt{2}, \sqrt{2 + \sqrt{2}}, \sqrt{2 + \sqrt{2 + \sqrt{2}}}, \ldots \).

   Use induction to show \( (x_n) \) is monotone and bounded. Deduce it is convergent and find its limit.

4. (i) Show that the sequence \( (a_n) \), \( a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \), is monotone increasing bounded above by \( 1 \).

   Deduce that it converges to a limit between \( \frac{1}{2} \) and 1.

   (ii) By expressing \( a_n \) in terms of partial sums of the harmonic series show that the limit above is \( \log 2 \).

5. Show that \( (n + 1)^n \geq 2n^n \), for all \( n \geq 1 \). Deuce that although \( (n + 1) \sim n \) as \( n \to \infty \) it is NOT true that \( (n + 1)^n \sim n^n \).

   Find a simple asymptotic estimate for \( (n + 1)^n \) in terms of \( n^n \).
6. Explore the possibility of summing these rearrangements of the alternating harmonic series.

(i) \[1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{7} - \frac{1}{9} + \frac{1}{11} + \frac{1}{13} - \frac{1}{15} - \frac{1}{17} + \cdots.\]

(ii) \[1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} + \frac{1}{15} - \frac{1}{17} + \cdots.\]

7. (The Cauchy Condensation Test)

(i) Observe that for \((a_n)\) a monotone decreasing of positive real numbers

\[\frac{1}{2}2^n a_{2^n} \leq \sum_{k=2^{n-1}+1}^{2^n} a_n \leq 2^{n-1}a_{2^n-1}.\]

Hence establish the Cauchy Condensation Test:

If \((a_n)\) is a monotone decreasing sequence of positive real numbers then the series \(\sum (a_n)\) and \(\sum a_{2^n}2^n\) either both converge or both diverge.

(ii) Use the Cauchy Condensation Test to show that \(\sum \frac{1}{n^p}\) is convergent if and only if \(p > 1\).

(iii) Use the Cauchy Condensation Test to show that \(\sum \frac{1}{n(\log n)^p}\) is convergent if and only if \(p > 1\).

8. (The Integral Test)

(i) Show that if \(a_n = f(n)\) for if \(f(x)\) is a positive decreasing function on 
\([1, \infty)\)

\[a_2 + \cdots + a_n \leq \int_1^n f(x) \, dx \leq a_1 + a_2 + \cdots + a_{n-1}\]

Hence establish the Integral Test:

Suppose that \(a_n = f(n)\) where \(f(x)\) is positive decreasing on \([1, \infty)\). Then the series \(\sum_{n=1}^{\infty} a_n\) is convergent if and only the integral

\[\int_1^{\infty} f(x) \, dx = \lim_{X \to \infty} \int_1^X f(x) \, dx\]

has a finite value.

(ii) Use the Integral Test to show that \(\sum \frac{1}{n(\log n)^p}\) is convergent if and only if \(p > 1\).