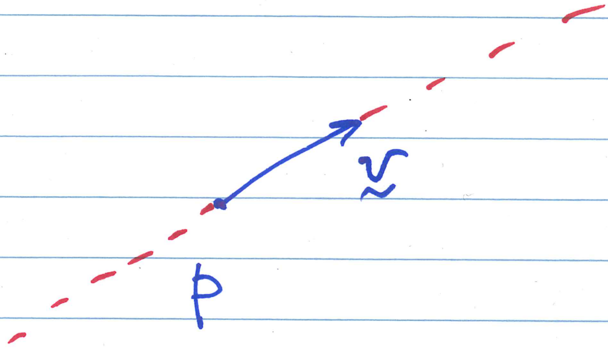
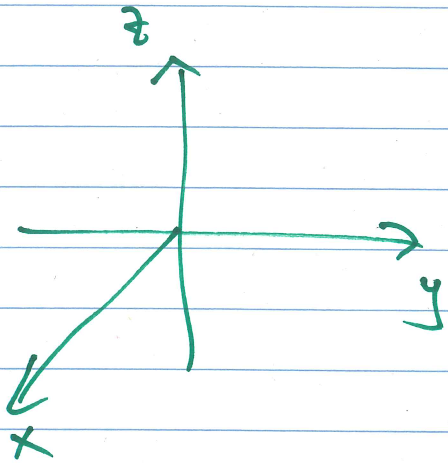
L. 10 Lines in Space





Eq-n of the line passing through $P = (x_0, y_0, z_0)$ in the direction of

$\underset{\sim}{v} = (a, b, c)$ given by:

$$\boxed{(x, y, z) = (x_0, y_0, z_0) + t \cdot (a, b, c)}$$

parametric form

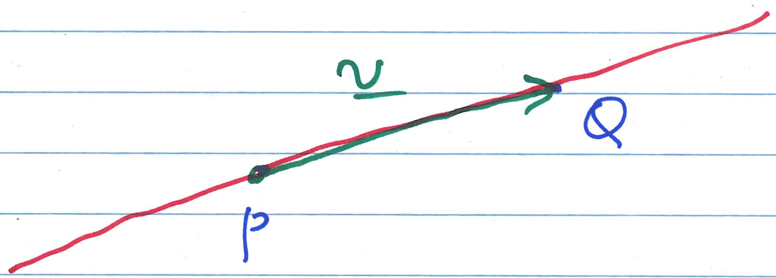
If $a \neq 0$, $b \neq 0$, $c \neq 0$:

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

Cartesian form

<u>Exercise:</u> Find equation of the line passing through the points
$P = (1,2,3)$ and $Q = (-1,2,4)$.

<u>Sol:</u>



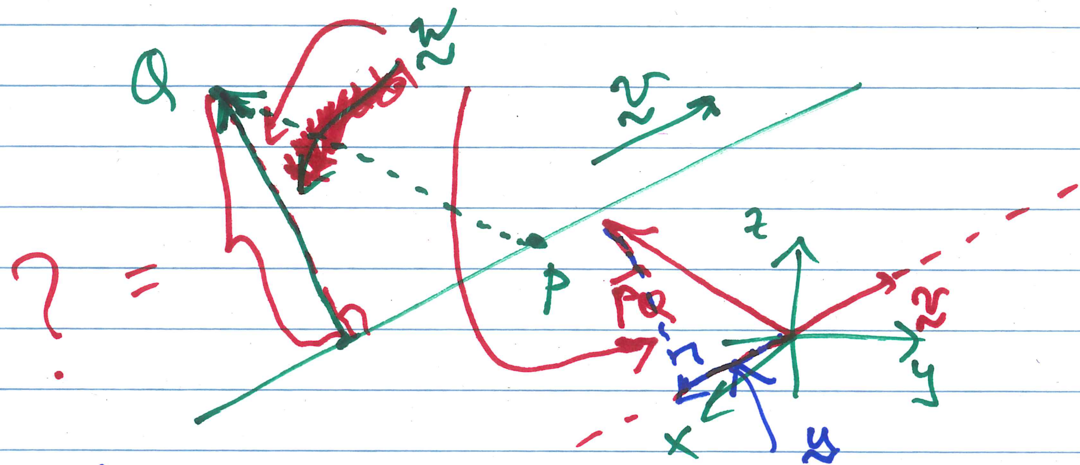
The direction of the line passing through P & Q coincides with the direction of $\overrightarrow{PQ}$.

$$\overrightarrow{PQ} = (-1,2,4) - (1,2,3) = (-2,0,1)$$

then the eq-n of the line passing through P & Q:

$$\boxed{(x,y,z) = (1,2,3) + t\,(-2,0,1) \quad \text{where } t \text{ runs over real numbers}}$$

Distance between a pt & a line



Question: What is the distance between pt. Q & the line passing through P in the direction of $\underset{\sim}{v}$?

Answer: Step 1: Find the orthogonal projection of $\overrightarrow{PQ}$ on the line through the origin generated by $\underset{\sim}{v}$.

Denote

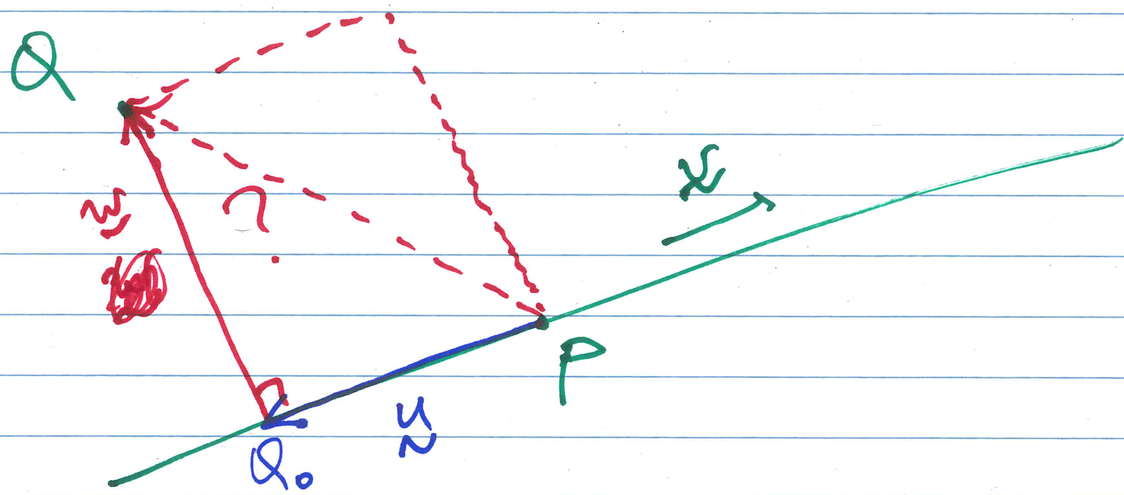
$$\underset{\sim}{u} = P_{\underset{\sim}{v}}(\overrightarrow{PQ}) = \frac{\overrightarrow{PQ} \cdot \underset{\sim}{v}}{|\underset{\sim}{v}|^2} \cdot \underset{\sim}{v}$$

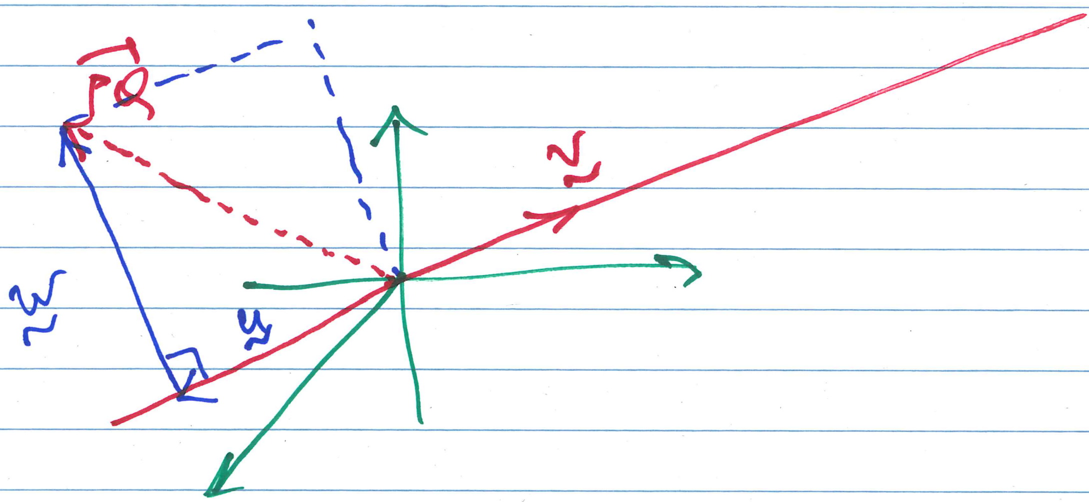
Step 2: Find the vector component of $\overrightarrow{PQ}$ orthogonal to the direction of $\underset{\sim}{v}$:

We call this vector $\underset{\sim}{w}$, & we know

$$\underset{\sim}{w} = \vec{PQ} - P_{\underset{\sim}{v}}(\vec{PQ}) = \vec{PQ} - \underset{\sim}{u}$$

$$= \vec{PQ} - \frac{\vec{PQ}\cdot\underset{\sim}{v}}{|\underset{\sim}{v}|^2}\cdot\underset{\sim}{v}$$





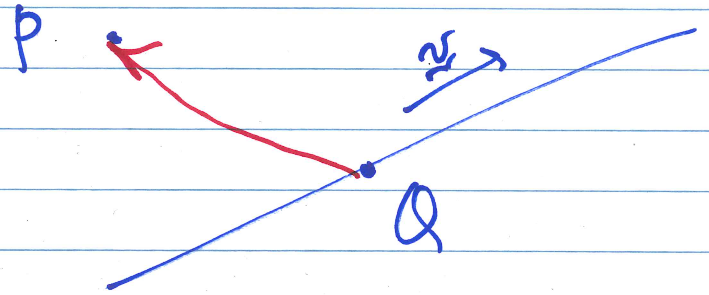
Task: To find the height of the rectangle generated by $\underset{\sim}{u}$ & $\underset{\sim}{w}$:

$$\Rightarrow \frac{\text{Area Rectangle}}{\text{length of a side}}$$

<u>Exercise:</u> Find the distance between the point $P = (1,2,3)$ and the line given by the eq-n:

$$\frac{x-3}{5} = \frac{y+1}{4} = \frac{z}{2}$$

<u>Sol:</u>



then $Q = (3,-1,0)$ on the line.

$$\Rightarrow \quad \overrightarrow{QP} = (1,2,3) - (3,-1,0) = (-2,3,3)$$

Our line is in the direction of

$$\underset{\sim}{v} = (5,4,2).$$

$$\Rightarrow \quad \text{distance} = \frac{|(-2,3,3) \times (5,4,2)|}{|(5,4,2)|} = \frac{\sqrt{6^2+19^2+23^2}}{\sqrt{5^2+4^2+2^2}}$$

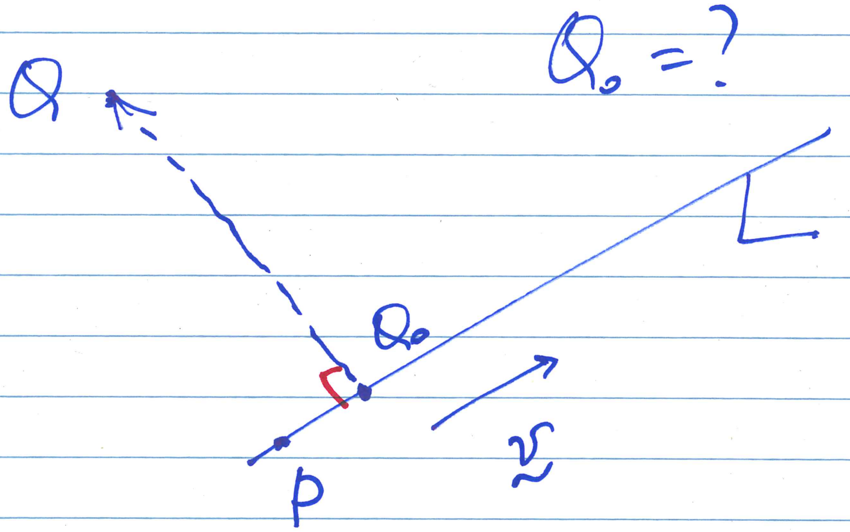
$$\begin{array}{ccccc} i & j & k & i & j \\ -2 & 3 & 3 & -2 & 3 \\ 5 & 4 & 2 & 5 & 4 \end{array} \qquad \Rightarrow \quad (-2,3,3) \times (5,4,2) = (6-12)i$$

$$+ (15+4)j + (-8-15)k = (-6,19,-23)$$

Exercise: Find the closest pt to $Q=(1,2,3)$ lying on the line L given by

$$x+1 = \frac{y-1}{2} = \frac{z-2}{4}$$

Sol:



Let

$Q_0 = (x, y, z)$ and it satisfies:

$$\overrightarrow{Q_0Q} \perp \underset{\sim}{v}$$

$$(1-x, 2-y, 3-z) \perp (1,2,4)$$

$$\Rightarrow \begin{cases} (1-x)\cdot 1 + (2-y)\cdot 2 + (3-z)\cdot 4 = 0 \\ x+1 = \frac{y-1}{2} = \frac{z-2}{4} \end{cases}$$