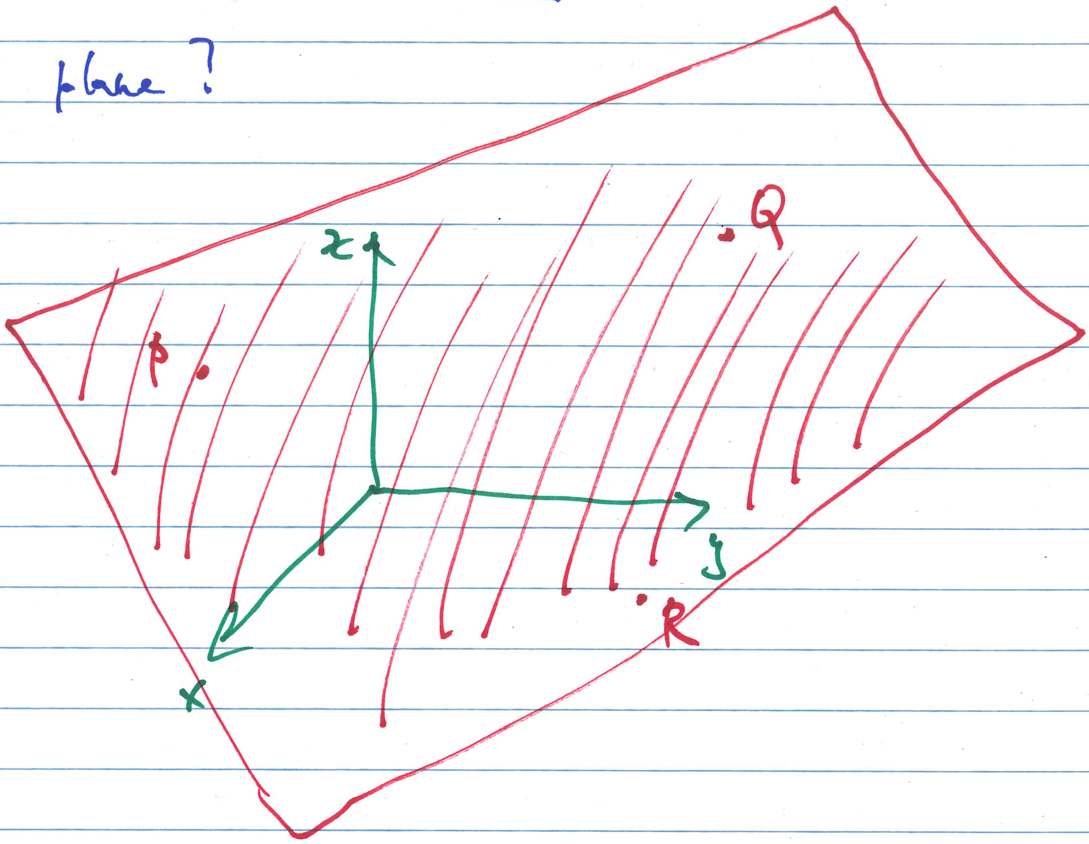
L.11

Planes in the Space

Q: What is a plane?

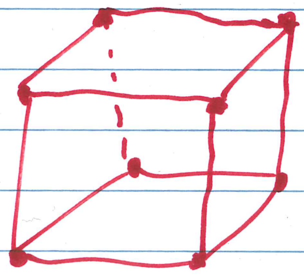
Intuitively:



Any 3 generic points in the space generate a plane.

Generate – there is a unique plane containing the given points.

Generic points – three (or more) points such that no three points lie on a single line.

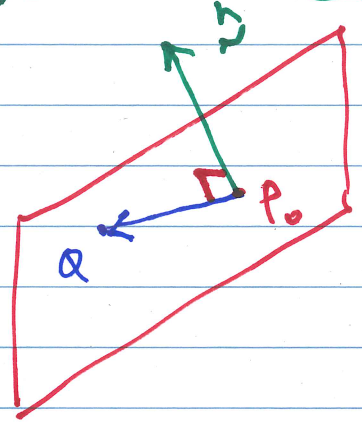


Configuration of 8 points in a generic position.

Formally:

A plane $\mathcal{P}$ is completely determined by two features:

- a point P_0 on $\mathcal{P}$
- a vector $\underset{\sim}{n}$ orthogonal to $\mathcal{P}$



Q is on $\mathcal{P}$ $\iff$ $\overrightarrow{P_0Q} \perp \underset{\sim}{n}$.

Ex: Find the eq-n of the plane containing the point $(1,2,3)$ and perpendicular to the vector $\underset{\sim}{v} = \underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$.

Sol:

$$P_0 = (1,2,3)$$

If $Q = (x,y,z)$ then $\overrightarrow{P_0Q} = (x-1, y-2, z-3)$

(x, y, z) is on the plane $\iff$

$$(x-1, y-2, z-3) \perp (1, -2, 3)$$

$\iff$

$$(x-1)\cdot 1 + (y-2)\cdot(-2) + (z-3)\cdot 3 = 0$$

$$-1 \qquad +4 \qquad -9$$

$\iff$

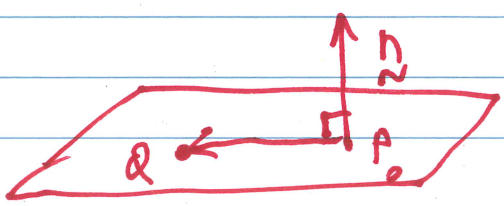
$$\boxed{1\cdot x + (-2)\cdot y + 3\cdot z = 6}$$

Rem: The coefficients of x, y, z are exactly the coordinates of the vector of the normal.

In general, assume $\underset{\sim}{n} = (a, b, c)$ & $P_0 = (x_0, y_0, z_0)$. Then $Q = (x, y, z)$ is on the plane perpendicular to $\underset{\sim}{n}$ and containing P_0

$\iff$

$$\overrightarrow{P_0 Q} = (x - x_0, y - y_0, z - z_0) \perp \underset{\sim}{n} = (a, b, c)$$



$\Leftrightarrow$

$$a \cdot (x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$-a \cdot x_0 \qquad -b \cdot y_0 \qquad -c \cdot z_0$$

$\Leftrightarrow$

$$\boxed{a \cdot x + b \cdot y + c \cdot z = d}$$

where

$$d = a \cdot x_0 + b \cdot y_0 + c \cdot z_0$$

Cartezian eq-n of the plane containing (x_0, y_0, z_0) & perpendicular to (a, b, c).

Ex: Whether the given points are on the plane given by the eq-n: $1 \cdot x - 2 \cdot y + 3 \cdot z = 2$?

(a) $(1, 3, 4)$

check: $1 \cdot 1 - 2 \cdot 3 + 3 \cdot 4 \overset{?}{=} 2$

No

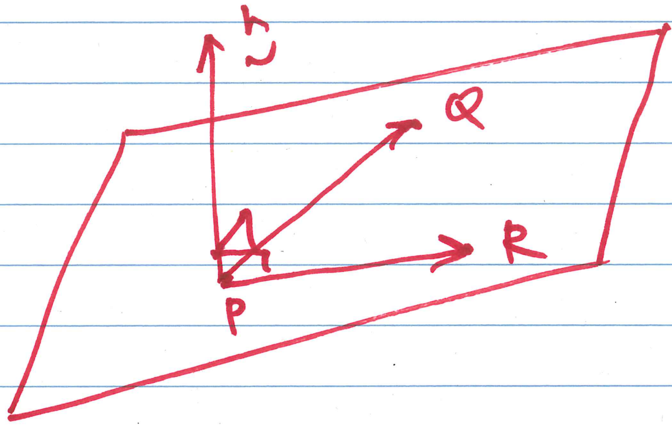
$\Rightarrow (1, 3, 4)$ is not on the plane

(b) ~~(1, 2, 4)~~ $(0, 0, \frac{2}{3})$

check: $1 \cdot 0 - 2 \cdot 0 + 3 \cdot \frac{2}{3} = 2$ Yes !!!

it is on the plane.

Task: How to find the eq-n of the plane generated by three generic points P, Q, R?



Answer: We find vector of normal

$\underset{\sim}{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$ and taking one of the points, for instance ... Q.

Ex: Find eq-n of the plane containing $(1,2,3)$, $(1,-1,2)$, $(2,1,0)$.

Sol: Denote by

$$P = (1,2,3), \quad Q = (1,-1,2), \quad R = (2,1,0).$$

Then

$$\overrightarrow{PQ} = (0,-3,-1)$$
$$\overrightarrow{PR} = (1,-1,-3)$$

next:

$\underset{\sim}{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$

& then we will find the eq-n.