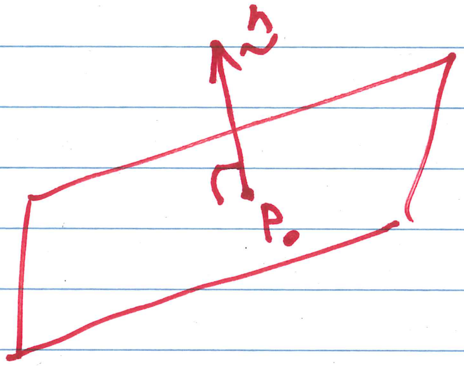
L.12

Planes in Space

Ex: Find an eq-n of the plane containing the points $(1,2,3)$, $(1,-1,2)$ and $(2,1,0)$.

Sol:

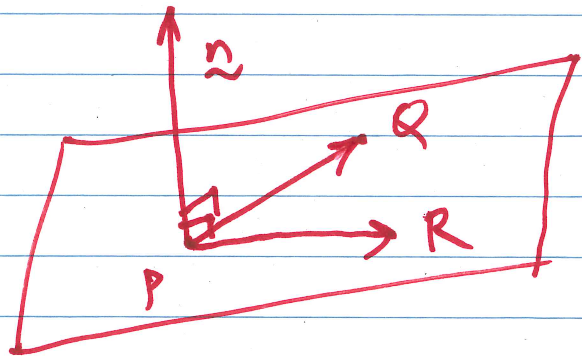
Recall,



an eq-n of the plane containing $P_0 = (x_0, y_0, z_0)$ and orthogonal to the vector $\underset{\sim}{n} = (a, b, c)$ is

$$\boxed{ax + by + cz = ax_0 + by_0 + cz_0}$$

In our case,



denote by

$P = (1,2,3)$

$Q = (1,-1,2)$

$R = (2,1,0)$

$\Rightarrow \quad \vec{PQ} = (0,-3,-1)$, $\vec{PR} = (1,-1,-3)$

Let $\underset{\sim}{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$. Then $\underset{\sim}{n} \perp \overrightarrow{PQ}, \overrightarrow{PR}$.

$$\begin{array}{ccccc} i & j & k & i & j \\ 0 & -3 & -1 & 0 & -3 \\ 1 & -1 & -3 & 1 & -1 \end{array}$$

$$\Rightarrow \quad \underset{\sim}{n} = (9-1)\underset{\sim}{i} + (-1-0)\underset{\sim}{j} + (0-(-3))\underset{\sim}{k}$$

$$= 8\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k} = (8,-1,3)$$

An eq-n of the plane containing $Q = (1,-1,2)$ and orthogonal to $\underset{\sim}{n} = (8,-1,3)$ is

$$8 \cdot x - y + 3z = 8 \cdot 1 + (-1)\cdot(-1) + 3 \cdot 2$$

$$\Rightarrow \quad \boxed{8x - y + 3z = 15}$$

Rem:

1) The eq-n $-24x + 3y - 9z = -45$ is equivalent to $8x - y + 3z = 15$. They have the same solutions.

2) We can check whether P, Q and R satisfy the eq-n: Plug $P = (1,2,3)$ into the eq-n:

$$8 \cdot 1 - 2 + 3 \cdot 3 \stackrel{?}{=} 15 \quad \text{true!}$$

$R = (2,1,0)$

$8 \cdot 2 - 1 + 3 \cdot 0 \stackrel{?}{=} 15$

yes!

Lines and Planes in Space - Geometry

Q: What is the intersection of:

A) two planes?

Generically, it is a line, but if two planes are parallel then it is either empty set or it is a plane.

Parallel planes - means the normal vectors are parallel.

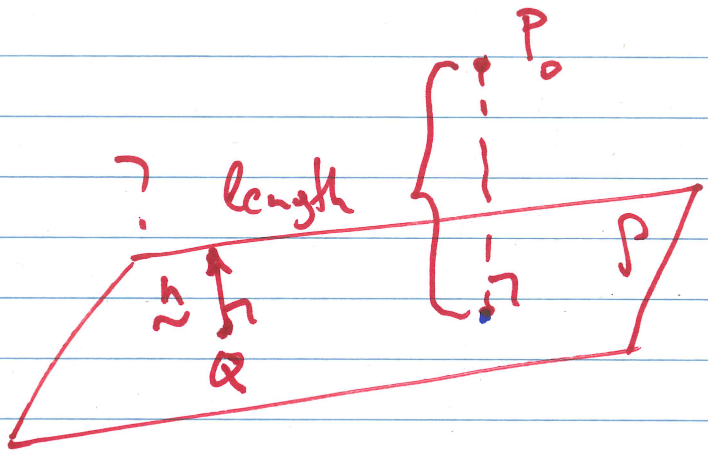
B) two lines?

Generically, empty set. But it might be a point (if they lie in the same plane), or it might be a line.

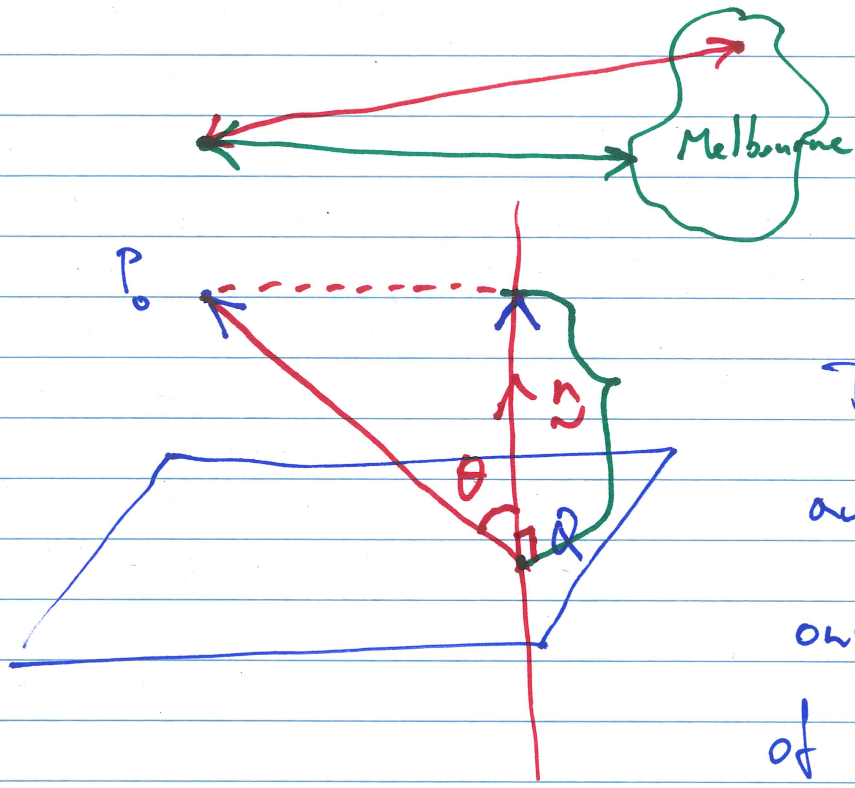
C) line and plane?

Generically, a point. But if the line is parallel to the plane (the direction of the line is orthogonal to the normal vector of the plane) then it is an empty set or it is the line.

Distance from a point to a plane



Task: Find the distance between point P_0 and the plane ρ containing Q and orthogonal to $\underset{\sim}{n}$.



Take $\overrightarrow{QP_0}$ and project onto the direction of $\underset{\sim}{n}$, and find the length of the projection

$$\text{Distance} = |\overrightarrow{QP_0}| \cdot \cos(\theta) = \frac{|\overrightarrow{QP_0}| \cdot |\underset{\sim}{n}| \cdot \cos\theta}{|\underset{\sim}{n}|} =$$

$$= \boxed{\frac{|\overrightarrow{QP_0} \cdot \underset{\sim}{n}|}{|\underset{\sim}{n}|}}$$

Ex: Find the distance between points $(1,2,3)$ and the plane ~~set~~ given by the eq-n $x - y + 2z = 3$.

Sol: First, let's find some point on the plane.

$Q = (0,0,z)$: $2z = 3 \Rightarrow z = \frac{3}{2}$.

$\Rightarrow (0,0,3/2)$ is on the plane.

$\Rightarrow P_0 = (1,2,3) \Rightarrow \overrightarrow{QP_0} = (1,2,\frac{3}{2})$

$$\Rightarrow \text{Distance} = \frac{|(1,2,\frac{3}{2}) \cdot (1,-1,2)|}{|(1,-1,2)|}$$

$$= \frac{|1-2+3|}{\sqrt{1^2+1^2+2^2}} = \frac{2}{\sqrt{6}}$$