L.13

Systems of Linear Equations

Q: What is a system of linear eq-ns?

Ex:

$$\begin{cases} 3x - y + z = 11 \\ -x \quad + z = 2 \end{cases} \quad (*)$$

System with 3 unknowns (x, y, z), and 2 eq-ns.

General Solution: All triples (x, y, z) which satisfy all the eq-ns.

Sometimes - a system can be inconsistent - no solutions at all.

Ex:

$$\begin{cases} x_1 - x_2 + x_3 - x_4 + x_5 = -7 \\ 2x_1 - 2x_2 + 2x_3 - 2x_4 + 2x_5 = -15 \end{cases}$$

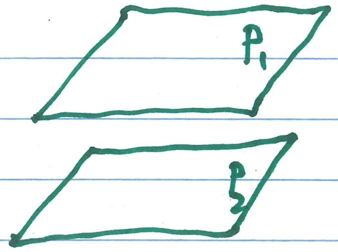
INCONSISTENT

Back to system $(*)$:

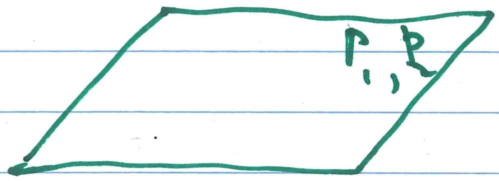
Every eq-n describes a plane in the space.

$\Rightarrow$ General solution of this system is the intersection of two planes.

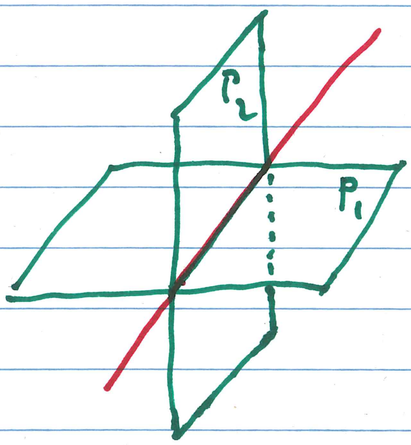
We can have three possibilities:

1) 

$\Rightarrow$ Inconsistent

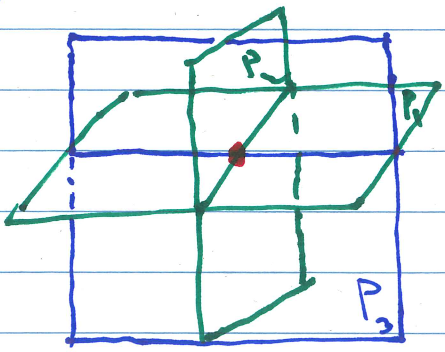
2) 

$\Rightarrow$ The general solution is a 2-dimensional plane

3) 

$\Rightarrow$ The general sol-n is a 1-dim. line

<u>Rem:</u> If we would have 3 eq-ns in three unknowns, then we could have also the following possibility:



$\Rightarrow$ The solution is a one point in the space $\Rightarrow$ Unique solution

This is a general fact:

<u>Theorem:</u> A system of linear eq-ns has either

(A) No solutions (Inconsistent)

(B) Exactly one solution (Unique sol-n)

(C) Infinitely many solutions

Back to ⊛:

$$\begin{cases} 3x - y + z = 11 \\ -x + z = 2 \end{cases}$$

The system ⊛ corresponds to the following <u>augmented matrix</u>:

$$\left(\begin{array}{ccc|c} 3 & -1 & 1 & 11 \\ -1 & 0 & 1 & 2 \end{array}\right)$$

$$\begin{cases} x - y = 3 \\ x + y = 2 \end{cases} \Leftrightarrow \begin{cases} x - y = 3 \\ 2y = -1 \end{cases}$$

$$\begin{array}{r} x + y = 2 \\ - \quad x - y = 3 \\ \hline 2y = -1 \end{array}$$

$$\Downarrow$$

$$y = -\frac{1}{2}$$

$$x = 3 + y = 3 - \frac{1}{2} = \frac{5}{2}$$

<u>Q:</u> How do we solve systems?

<u>Idea:</u> We <u>add/deduct</u> different eq-ns in the system, and obtain <u>equivalent</u> systems which are easier to solve.

The systems are <u>equivalent</u> if they have the same general solutions.

In our case, (*) is <u>equivalent</u> to

$$\begin{cases} 3x - y + z = 11 \\ -\frac{1}{3}y + \left(1+\frac{1}{3}\right)z = 2 + \frac{1}{3}\cdot 11 \end{cases}$$

In other words:

$$\begin{cases} 3x - y + z = 11 \\ -\frac{1}{3}y + \frac{4}{3}z = \frac{17}{3} \end{cases} \qquad (**)$$

In matrix form:

$$\left(\begin{array}{ccc|c} 3 & -1 & 1 & 11 \\ -1 & 0 & 1 & 2 \end{array}\right) \xrightarrow{R_2 \leftarrow R_2 + \frac{1}{3}R_1} \left(\begin{array}{ccc|c} 3 & -1 & 1 & 11 \\ 0 & -\frac{1}{3} & \frac{4}{3} & \frac{17}{3} \end{array}\right)$$

Notice that (**) is easy to solve:

For every value of z corresponds unique pair of x & y. Let $z = t$ $\Rightarrow$ from second eq-n:

$-\frac{1}{3}y = \frac{17}{3} - \frac{4}{3}t \Rightarrow y = -17 + 4t$

From the first eq-n: $3x = 11 + y - z = 11 + (-17+4t) - t$
$= -6 + 3t$

$\Rightarrow x = -2 + t$

$\Rightarrow$ The general solution of ⊛ is:

$$\left\{ \begin{bmatrix} -2 \\ -17 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

in — real numbers

$$= \left\{ \begin{bmatrix} -2+t \\ -17+4t \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

$\Rightarrow$ The general sol-n is a line.

t called a free parameter.