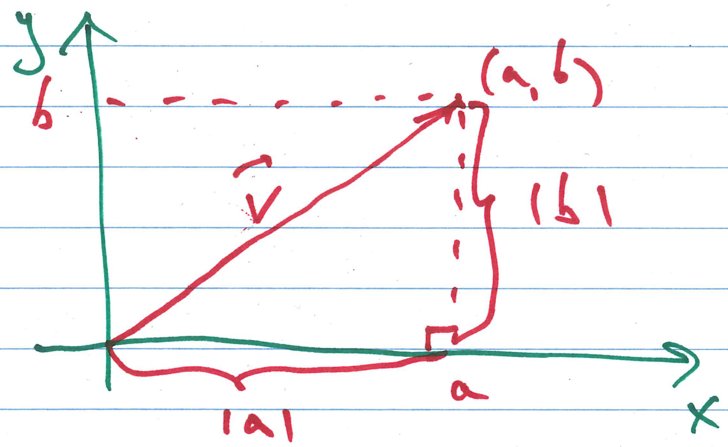
L.3

Q: Given a vector $\vec{v} = (a, b)$ in the plane, how can we find its length?

Answer:



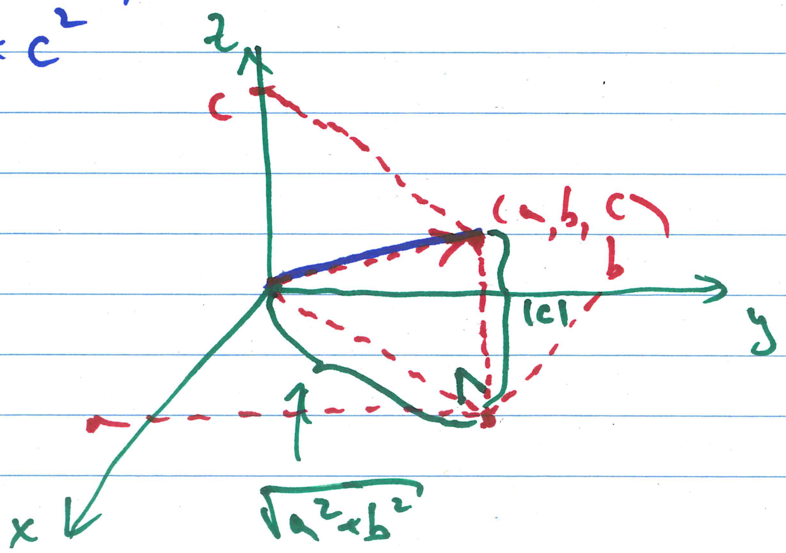
by Pythagoras:

$$|\vec{v}| = \sqrt{|a|^2 + |b|^2} = \sqrt{a^2 + b^2}$$

the length of $\vec{v}$

Similarly, given $\vec{u} = (a, b, c)$ in 3-d space,

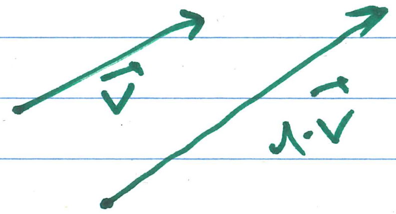
$$|\vec{u}| = \sqrt{a^2 + b^2 + c^2}$$



Q: How magnitude (length) of a vector changed if ~~t~~ the vector is multiplied by a scalar λ (real #)?

Answer:

$$|\lambda\cdot\vec{v}| = |\lambda|\cdot|\vec{v}|$$



$$\lambda\cdot(a,b) = (\lambda a, \lambda b)$$

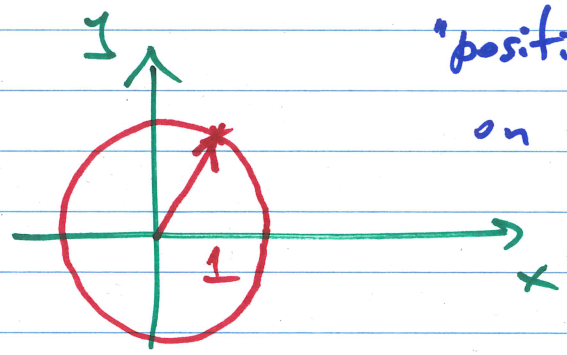
$$|(\lambda a, \lambda b)| = \sqrt{(\lambda a)^2+(\lambda b)^2}$$

$$= |\lambda|\cdot\sqrt{a^2+b^2}$$

$$= |\lambda|\cdot|(a,b)|$$

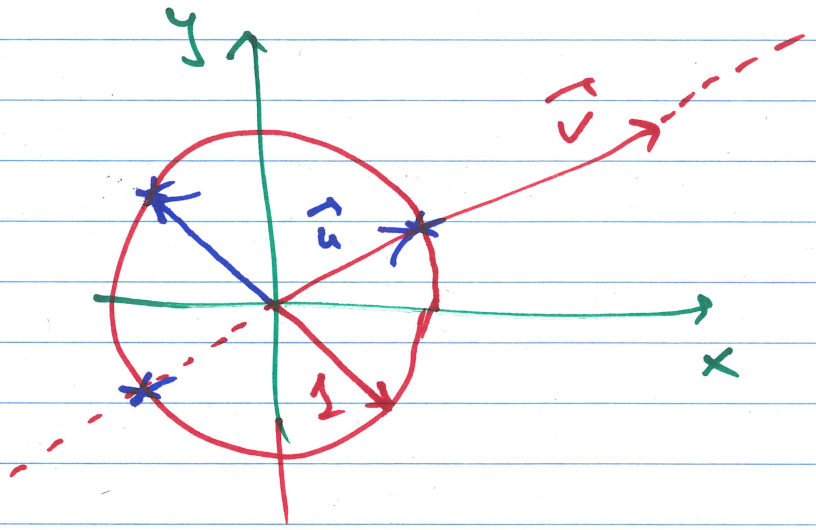
Unit vector – vector of length one.

In the plane – unit vectors = all "position vectors" on the unit circle around the origin.



Q: How can we obtain from a (non-zero) vector $\vec{v}$, another vector $\vec{u}$ of length one & parallel to $\vec{v}$?

A:



$$\vec{u} = \lambda \cdot \vec{v}$$

s.t.

$$|\vec{u}| = |\lambda| \cdot |\vec{v}| = 1$$

$$|\lambda| = \frac{1}{|\vec{v}|} \Rightarrow \vec{u} = \frac{1}{|\vec{v}|} \cdot \vec{v}$$

Def (parallel vectors):

$\vec{u}$ & $\vec{v}$ are parallel if there exists λ in $\mathbb{R}$ ← (the set of real numbers) such that

$$\vec{u} = \lambda \vec{v} \quad \text{or} \quad \vec{v} = \lambda \vec{u}$$

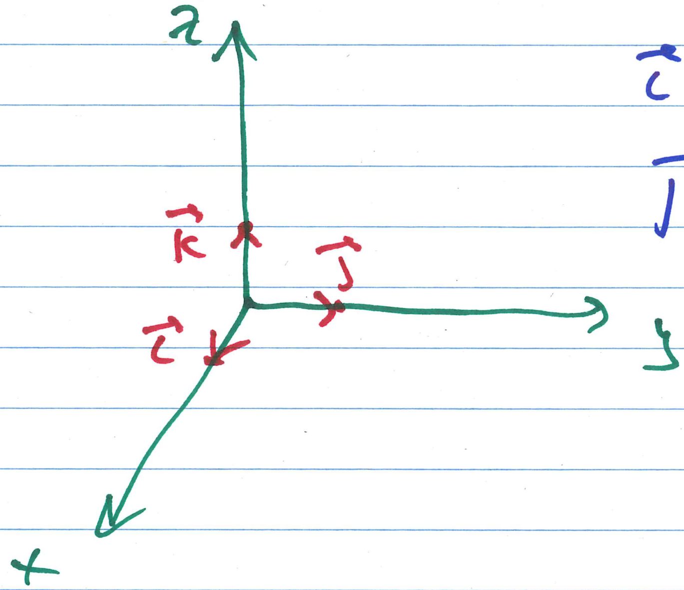
Claim: $\vec{0}$ is parallel to any other vector.

Pf: Take any $\vec{v}$, then $\vec{0} = 0 \cdot \vec{v}$

Exercise: Let $\vec{v} = 5\underset{\sim}{\vec{i}} + 3\underset{\sim}{\vec{j}}$, $\vec{w} = -\underset{\sim}{\vec{i}} + \underset{\sim}{\vec{k}}$.

Find $|\vec{v} + \vec{w}|$.

Sol:



$\vec{i} = (1,0,0)$

$\vec{j} = (0,1,0)$

$\vec{k} = (0,0,1)$

In other words, any vector

$$(a,b,c) = (a,0,0) + (0,b,0) + (0,0,c)$$

$$= a\cdot(1,0,0) + b(0,1,0) + c\cdot(0,0,1) =$$

$$a\cdot\vec{i} + b\cdot\vec{j} + c\cdot\vec{k}$$

Similarly, $(a,b) = a\cdot\vec{i} + b\cdot\vec{j}$

where $\vec{i} = (1,0)$, $\vec{j} = (0,1)$.

Back to exercise: $\vec{v} = (5,3,0)$, $\vec{w} = (-1,0,1)$

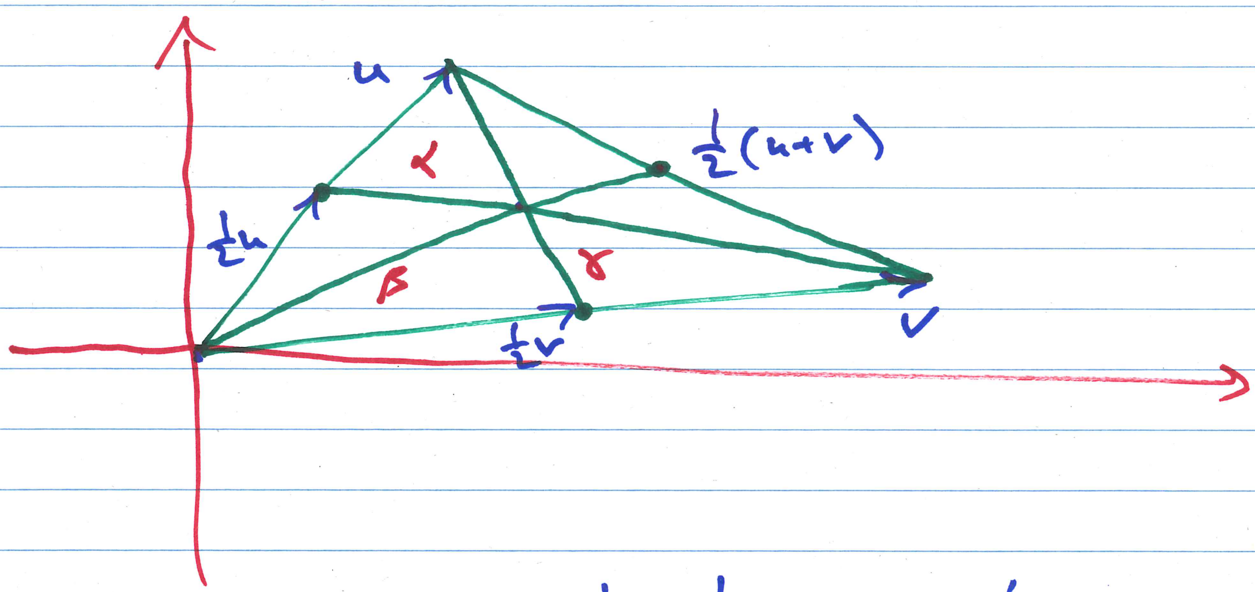
$$\Rightarrow \vec{v} + \vec{w} = (5,3,0) + (-1,0,1) = (4,3,1)$$

$$\Rightarrow |\vec{v} + \vec{w}| = \sqrt{4^2 + 3^2 + 1^2} = \sqrt{26}$$

Our exercise will motivate the notion of linear independence (of vectors)

Exercise: Show that all medians of a triangle intersect at one point.

Sol:



All the medians intersect at one pt $\iff$ There exist α, β, γ real numbers such that

$$\frac{1}{2}u + \alpha \cdot \left(v - \frac{1}{2}u\right) =$$

$$\frac{1}{2}v + \gamma \cdot \left(u - \frac{1}{2}v\right) =$$

$$\beta \cdot \frac{1}{2} \cdot (u+v)$$

Q: Can we find α, β, γ real s.t.

$$\frac{1}{2}u + \alpha\left(v - \frac{1}{2}u\right) = \frac{1}{2}v + \gamma\left(u - \frac{1}{2}v\right) = \beta \cdot \frac{1}{2}(u+v)?$$

How can we solve it ???!!!

Let's start ...

$$\frac{1}{2}u + \alpha\left(v - \frac{1}{2}u\right) = \frac{1}{2}v + \gamma\left(u - \frac{1}{2}v\right)$$

$$\left(\frac{1}{2} - \frac{\alpha}{2} - \gamma\right)\cdot\vec{u} + \left(\alpha - \frac{1}{2} + \frac{\gamma}{2}\right)\vec{v} = \vec{0}$$

If one of the "coefficients" is non-zero then $\vec{u} \parallel \vec{v}$.

$$\Rightarrow \begin{cases} \frac{1}{2} - \frac{\alpha}{2} - \gamma = 0 \\ \alpha - \frac{1}{2} + \frac{\gamma}{2} = 0 \end{cases}$$

$$\frac{1}{2} - \frac{3}{2}\gamma = 0 \quad \Rightarrow \quad \gamma = \frac{1}{3}$$

add twice the first eq-n plus the second eq-n

$$\alpha - \frac{1}{2} + \frac{1}{3}\cdot\frac{1}{2} = 0 \quad \Rightarrow \quad \alpha = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$$

To be continue....