L.5

We define the dot-product of two vectors $\vec{v}$ & $\vec{w}$ to be
$$\vec{v} = (a,b) \qquad \vec{w} = (c,d)$$

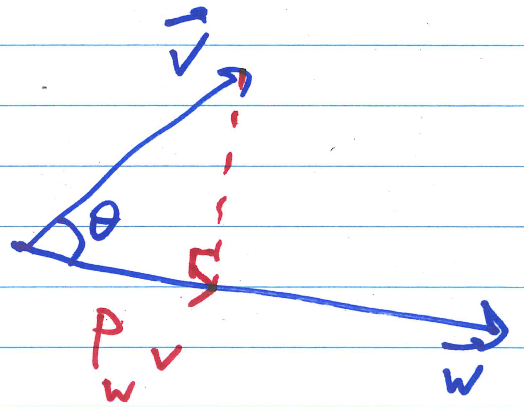
$$\boxed{\vec{v}\cdot\vec{w} = a\cdot c + b\cdot d}$$

We showed:

The projection of $\vec{v}$ on $\vec{w}$ can be expressed through the dot-product:

$$\boxed{P_{\vec{w}}\vec{v} = \frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\cdot\vec{w}}$$

Another way to compute the projection



The length of $P_w v$:

$$|P_w v| = |\vec{v}|\cdot\cos(\theta)$$

The direction of $P_w v$:

$$\frac{\vec{w}}{|\vec{w}|}$$

<u>Conclusion</u>: $$\boxed{P_{\vec{w}} \vec{v} = |\vec{v}| \cdot \cos(\theta) \cdot \frac{\vec{w}}{|\vec{w}|}}$$

<u>Corollary</u>: $$\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2} \cdot \vec{w} = \frac{|\vec{v}| \cdot \cos(\theta)}{|\vec{w}|} \cdot \vec{w}$$

$$\Rightarrow \quad \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2} = \frac{|\vec{v}| \cdot \cos(\theta)}{|\vec{w}|}$$

$$\Rightarrow \quad \boxed{\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cdot \cos(\theta)}$$

<u>Thm</u> (Cauchy-Schwartz inequality)

$$|\vec{v} \cdot \vec{w}| \leq |\vec{v}| \cdot |\vec{w}|,$$

and the equality is attained only if $\vec{v} \parallel \vec{w}$.

<u>Cor</u>: Two vectors $\vec{v}$ & $\vec{w}$ are <u>orthogonal</u> iff $\vec{v} \cdot \vec{w} = 0$.

Properties of dot-product

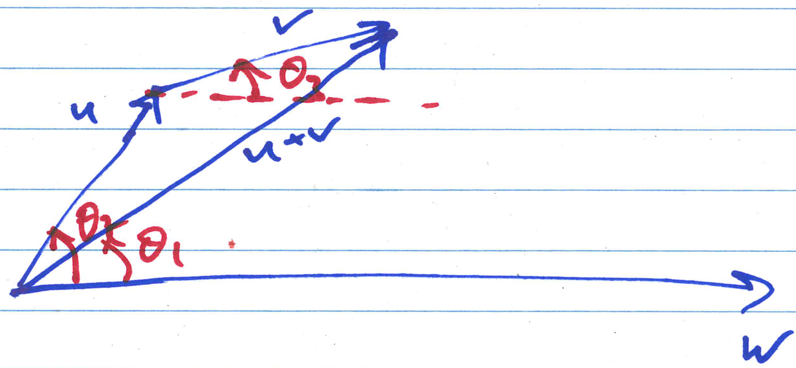
1) commutativity:

$$\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$$ clear from the def-n.

2) distributive law:

$$(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}$$

1^{st} Attempt:

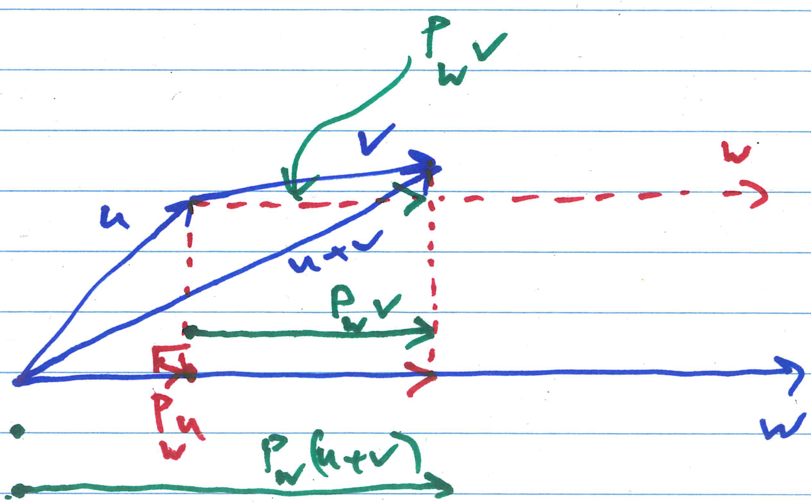


$LHS = |\vec{u}+\vec{v}|\cdot|\vec{w}|\cdot\cos(\theta_1)$

$RHS = |\vec{u}|\cdot|\vec{w}|\cdot\cos(\theta_2) + |\vec{v}|\cdot|\vec{w}|\cdot\cos(\theta_3)$

what to do ?!!

2^{nd} Attempt:



What is the connection between

$P_w u$, $P_w v$, $P_w(u+v)$.

$\Rightarrow$ We just showed:

$$\boxed{P_w(u+v) = P_w u + P_w v}$$

Recall, $$P_w v = \frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\cdot\vec{w}$$

$$P_w u = \frac{\vec{u}\cdot\vec{w}}{|\vec{w}|^2}\cdot\vec{w}$$

$$P_w(u+v) = \frac{(\vec{u}+\vec{v})\cdot\vec{w}}{|\vec{w}|^2}\cdot\vec{w}$$

$$\Rightarrow \underline{\frac{(\vec{u}+\vec{v})\cdot\vec{w}}{|\vec{w}|^2}}\cdot\vec{w} = \frac{\vec{u}\cdot\vec{w}}{|\vec{w}|^2}\cdot\vec{w} + \frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\cdot\vec{w}$$

$$= \left(\frac{\vec{u}\cdot\vec{w}}{|\vec{w}|^2} + \frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\right)\cdot\vec{w}$$

$$= \underline{\frac{\vec{u}\cdot\vec{w}+\vec{v}\cdot\vec{w}}{|\vec{w}|^2}}\cdot\vec{w}$$

$$\Rightarrow \boxed{(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w}+\vec{v}\cdot\vec{w}}$$

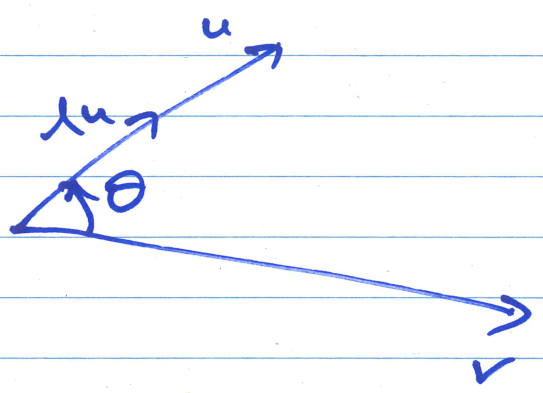
3) $(\lambda \cdot \bar{u}) \cdot \bar{v} = \lambda (\bar{u} \cdot \bar{v})$

for real # λ

Pf for $\lambda > 0$:

LHS $= (\lambda |\bar{u}| \cdot |\bar{v}| \cdot \cos(\theta)) = \lambda |\bar{u}| \cdot |\bar{v}| \cdot \cos(\theta)$

if $\lambda > 0$

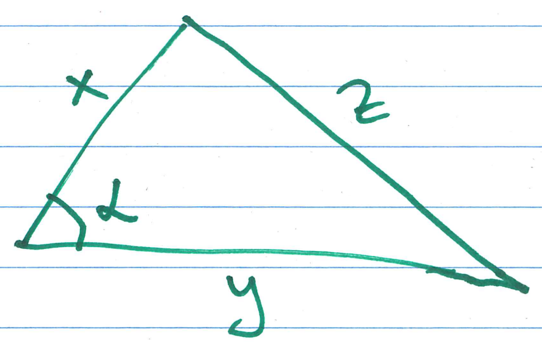


RHS $= \lambda \cdot |\bar{u}| \cdot |\bar{v}| \cdot \cos(\theta)$

Indeed, LHS = RHS.

4) $\bar{v} \cdot \bar{v} = |\bar{v}|^2$

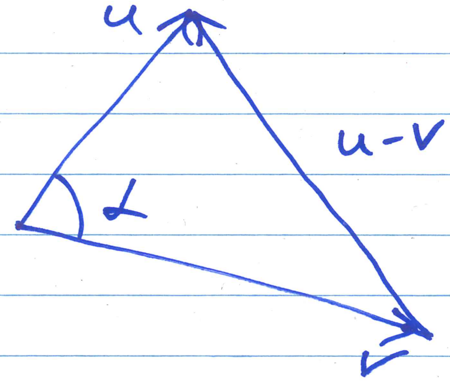
Exercise: Prove that for the triangle



we have the following identity:

$$z^2 = x^2 + y^2 - 2xy\cos(\alpha) \quad \left(\begin{array}{c}\text{cosine}\\ \text{rule}\end{array}\right)$$

d:



$x = |\bar{u}|, \quad y = |\bar{v}|, \quad z = |\bar{u} - \bar{v}|$

$z^2 = |\bar{u} - \bar{v}|^2 = (\bar{u} - \bar{v}) \cdot (\bar{u} - \bar{v}) =$

$= (\bar{u} - \bar{v}) \cdot \bar{u} - (\bar{u} - \bar{v}) \cdot \bar{v} =$

$= \bar{u} \cdot \bar{u} - \bar{v} \cdot \bar{u} - \bar{u} \cdot \bar{v} + \bar{v} \cdot \bar{v}$

$= |\bar{u}|^2 - 2 \cdot \bar{u} \cdot \bar{v} + |\bar{v}|^2$

$= x^2 + y^2 - 2 \cdot |\bar{u}| \cdot |\bar{v}| \cdot \cos(\alpha)$

$= x^2 + y^2 - 2 \cdot x \cdot y \cdot \cos(\alpha)$