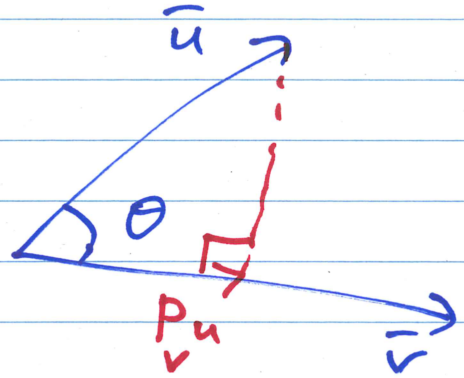
L. 6

Recall, for two vectors $\bar{u}, \bar{v}$ we defined their dot-product $\bar{u} \cdot \bar{v}$.

"Two definitions":

1) $$\bar{u} \cdot \bar{v} = |\bar{u}| \cdot |\bar{v}| \cdot \cos(\theta)$$



2) $$P_{\bar{v}} \bar{u} = \frac{\bar{u} \cdot \bar{v}}{|\bar{v}|^2} \cdot \bar{v}$$

Properties:

1) $\bar{u} \cdot \bar{v} = \bar{v} \cdot \bar{u}$ commutativity

2) $\bar{u} \cdot (\bar{v} + \bar{w}) = \bar{u} \cdot \bar{v} + \bar{u} \cdot \bar{w}$ distributive law

3) $(\lambda \bar{u}) \cdot \bar{v} = \lambda \cdot (\bar{u} \cdot \bar{v})$, λ - real #.

4) $\bar{u} \cdot \bar{u} = |\bar{u}|^2$.

In the plane:

$\bar{u} = (a, b)$; $\bar{v} = (c, d)$

Then $\bar{u} \cdot \bar{v} \doteq a \cdot c + b \cdot d$

In 3-d space:

$\vec{u} = x_1 \vec{i} + y_1 \vec{j} + z_1 \vec{k} \quad (= (x_1, y_1, z_1))$

$\vec{v} = x_2 \cdot \vec{i} + y_2 \cdot \vec{j} + z_2 \vec{k} \quad (= (x_2, y_2, z_2))$

then

$$\vec{u} \cdot \vec{v} \stackrel{\text{def}}{=\!=} x_1 x_2 + y_1 y_2 + z_1 \cdot z_2$$

<u>Exercise:</u> Let $\vec{u} = 5\vec{i} + \vec{j}$

$\bar{v} = \vec{j} - \vec{k}$

$\bar{w} = \vec{i} + \vec{j} + \vec{k}$

a) Find $\bar{u} \cdot \bar{v}$, $\bar{u} \cdot (\bar{v} + \bar{w})$

b) Determine whether the angles between $\bar{u}$ & $\bar{v}$, $\bar{v}$ & $\bar{w}$, and $\bar{u}$ & $\bar{w}$ are <u>acute</u> or <u>obtuse</u>

c) Find the scalar & vector component of $\bar{u}$ in the direction of $\bar{v}$ & in the direction orthogonal to $\bar{v}$.

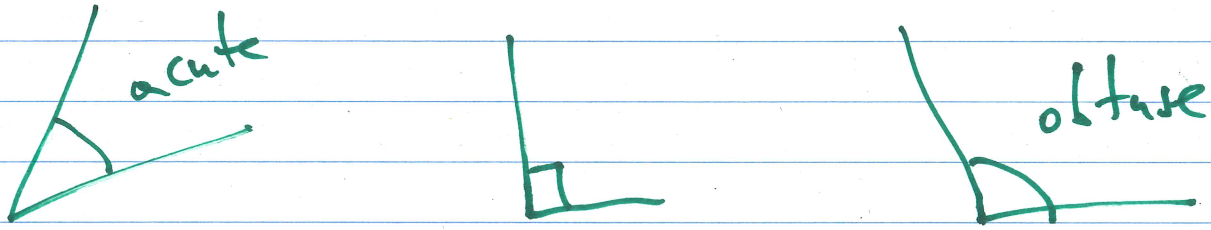
Sol:

a) $\bar{u}\cdot\bar{v} = 5\cdot 0 + 1\cdot 1 + 0\cdot(-1) = 1$

$\bar{u}\cdot(\bar{v}+\bar{w}) = \bar{u}\cdot(\bar{i}+2\bar{j}) = (5\bar{i}+\bar{j})\cdot(\bar{i}+2\bar{j})$

$= 5\cdot 1 + 1\cdot 2 + 0\cdot 0 = 7$

b)



cos(acute angle) > 0

cos(obtuse angle) < 0

Since $\bar{u}\cdot\bar{v} = \underbrace{|\bar{u}|}_{\geq 0}\cdot\underbrace{|\bar{v}|}_{\geq 0}\cdot\cos(\theta)$

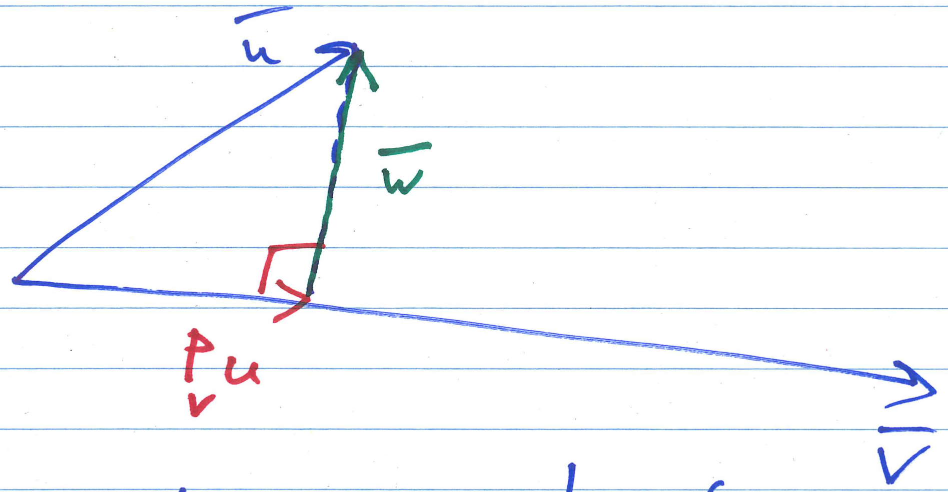
Thus, The angle between two vectors is acute $\Leftrightarrow$ their dot-product is positive

The angle between two vectors is obtuse $\Leftrightarrow$ their dot-product is negative

$\bar{u}\cdot\bar{v} = 1 > 0 \Rightarrow$ angle between $\bar{u}$ & $\bar{v}$ is acute.

In similar fashion, compute $\bar{u}\cdot\bar{w}$ & $\bar{v}\cdot\bar{w}$ ....

c)



$P_v u$ – is the component of $\bar u$ in the direction of $\bar v$.

$\bar w$ – is the component of $\bar u$ in the direction orthogonal to $\bar v$.

The following identity holds:

$$\bar u = P_{v}\bar u + \bar w$$

To find scalar & vector components of $\bar u$ in the direction of $\bar v$ ($\perp$ to $\bar v$) we just need to find:

scalar vector components of u the direction $\perp$ v.

$$P_v \bar u = \boxed{\lambda}\cdot\boxed{\frac{\bar v}{|\bar v|}} \qquad \bar w = \boxed{\mu}\cdot\boxed{\frac{\bar w}{|\bar w|}}$$

scalar vector components of u in the direction of v.

$$P_{v} u = \frac{\bar u \cdot \bar v}{|\bar v|^2}\cdot \bar v = \underbrace{\boxed{\frac{\bar u \cdot \bar v}{|\bar v|}}}_{\text{scalar component}} \cdot \underbrace{\boxed{\frac{\bar v}{|\bar v|}}}_{\text{vector component}}$$

$$\Rightarrow \quad \frac{\bar u \cdot \bar v}{|\bar v|} = \frac{1}{|\bar j - \bar k|} = \frac{1}{\sqrt{1^2+(-1)^2}} = \frac{1}{\sqrt{2}}$$

$$\frac{\bar v}{|\bar v|} = \frac{\bar j - \bar k}{\sqrt{2}} = \frac{1}{\sqrt{2}}\bar j - \frac{1}{\sqrt{2}}\bar k = \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

To find $\bar w$, notice:

$$\bar w = \bar u - P_{\bar v}\bar u = \underbrace{5\bar i + \bar j}_{\bar u} - \underbrace{\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\bar j - \frac{1}{\sqrt{2}}\bar k\right)}_{P_{\bar v}\bar u}$$

$$= (5\bar i + \bar j) - \left(\frac{1}{2}\bar j - \frac{1}{2}\bar k\right) =$$

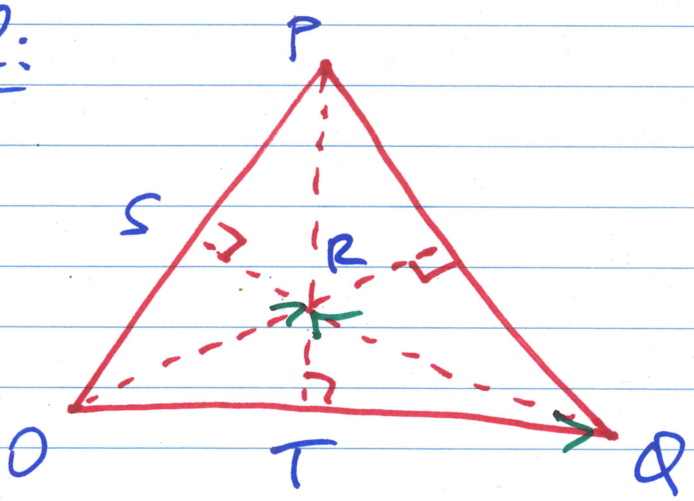
$$\cancel{= \frac{9}{2}\bar i + \bar j + \frac{1}{2}\bar k} \quad 5\bar i + \frac{1}{2}\bar j + \frac{1}{2}\bar k$$

$$= \underbrace{\boxed{\sqrt{5^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}}}_{\text{scalar component}} \cdot \underbrace{\boxed{\frac{5\bar i + \frac{1}{2}\bar j + \frac{1}{2}\bar k}{\sqrt{25\frac{1}{2}}}}}_{\text{vector component}}$$

of $\bar u$ orthogonal to $\bar v$.

Exercise: Prove that all altitudes in a triangle intersect in one point.

Sol:



R is the intersection pt of altitudes SQ & PT.

Enough to show: $\vec{OR} \perp \vec{PQ}$.
"orthogonal".

Therefore we have to show:

$$\vec{OR} \cdot \vec{PQ} = 0.$$

$$\vec{OR} \cdot \overline{PQ} = \left(\vec{OQ} + \vec{QR}\right) \cdot \left(\vec{OQ} - \vec{OP}\right)$$

$$= \overline{OQ} \cdot \overline{OQ} - \overline{OQ} \cdot \overline{OP} + \overline{QR} \cdot \overline{OQ} - \cancelto{0}{\overline{QR} \cdot \overline{OP}}$$

$$= \overline{OQ} \cdot \overline{OQ} - \overline{OQ} \cdot \overline{OP} + \overline{QR} \cdot \overline{OQ}$$

$$= \underbrace{(\vec{OQ} + \overline{QR})}_{\overline{OR}} \cdot \overline{OQ} - \overline{OQ} \cdot \overline{OP} = (\overline{OR} - \overline{OP}) \cdot \overline{OQ}$$

$$= \cancel{PO}\ \overline{PR} \cdot \overline{OQ} = 0$$