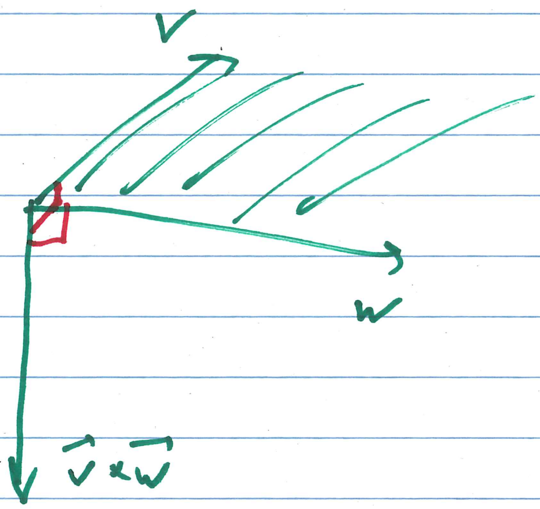
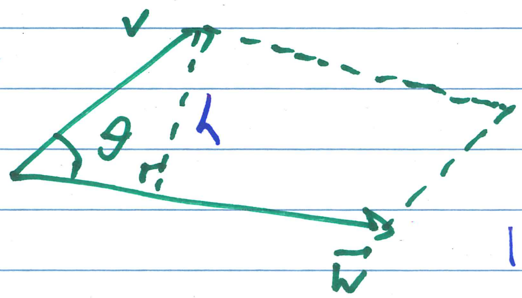
1.7 Cross Products

Given two vectors in 3-dim. space, say $\vec{v}$ & $\vec{w}$, we can define the vector $\vec{v} \times \vec{w}$ satisfying:



a) $\vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ & $\vec{w}$.

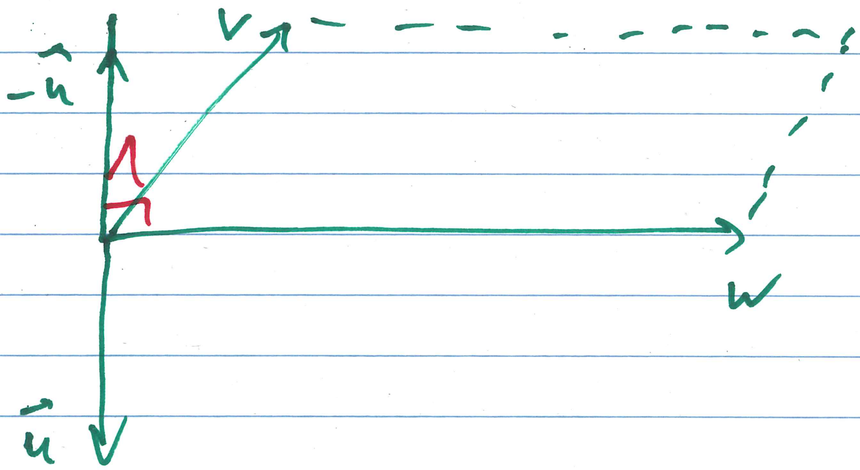
b) the length of $\vec{v} \times \vec{w}$ is equal to the area of the parallelogram formed by $\vec{v}$ & $\vec{w}$:



$$|\vec{v} \times \vec{w}| = |\vec{w}| \cdot h$$
$$= |\vec{w}| \cdot |\vec{v}| \cdot \sin(\theta)$$

Important: Cross product is **not defined** in the plane !!!

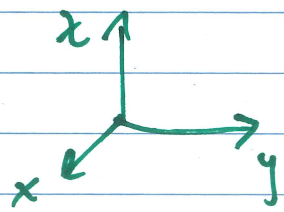
Q: There are two perpendicular directions to a given plane (up ↑, down ↓), which one do we choose?



The rule of Right Hand (wine bottle opener) tells us that $\vec{v} \times \vec{w}$ has the same direction as $\vec{u}$.

Conclusion: The direction of $\vec{w} \times \vec{v}$ is $-\vec{u}$.

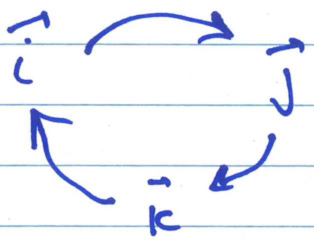
Rem:



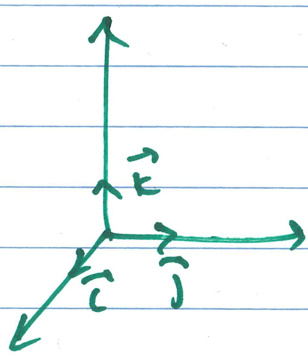
Q: How do we compute $\vec{v} \times \vec{w}$ efficiently?

A:

1) First, we remember:



Thus,
$$\begin{cases} \vec{i} \times \vec{j} = \vec{k} \\ \vec{j} \times \vec{k} = \vec{i} \\ \vec{k} \times \vec{i} = \vec{j} \end{cases}$$



$$\Rightarrow \begin{cases} \vec{j} \times \vec{i} = -\vec{k} \\ \vec{k} \times \vec{j} = -\vec{i} \\ \vec{i} \times \vec{k} = -\vec{j} \end{cases}$$

$$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}.$$

Secondly, we make use of "nice" algebraic properties of cross product:

For example,

$$\vec v = (1,2,3)$$

$$\vec w = (3,2,1)$$

Then

$$(\vec i + 2\vec j + 3\vec k)\times(3\vec i + 2\vec j + \vec k)$$

$$= \overset{\vec 0}{\cancel{1\cdot 3\,\vec i\times\vec i}} + 1\cdot 2\,\vec i\times\vec j + 1\cdot 1\cdot\vec i\times\vec k$$

$$+ 2\cdot 3\,\vec j\times\vec i + \overset{\vec 0}{\cancel{2\cdot 2\cdot\vec j\times\vec j}} + 2\cdot 1\cdot\vec j\times\vec k$$

$$+ 3\cdot 3\cdot\vec k\times\vec i + 3\cdot 2\cdot\vec k\times\vec j + \overset{\vec 0}{\cancel{3\cdot 1\cdot\vec k\times\vec k}}$$

$$= (2-6)\vec k + (9-1)\vec j + (2-6)\vec i$$

$$= -4\vec i + 8\vec j - 4\vec k = (-4,8,-4)$$

<u>Alternatively</u>:

$$\begin{array}{ccccc} i & j & k & i & j \\ 1 & 2 & 3 & 1 & 2 \\ 3 & 2 & 1 & 3 & 2 \end{array}$$

$$\vec v\times\vec w = 1\cdot 2\cdot\vec i + 3\cdot 3\cdot\vec j + 2\cdot 1\cdot\vec k - (3\cdot 2\cdot\vec k + 2\cdot 3\,\vec i + 1\cdot 1\cdot\vec j) = -4\vec i + 8\vec j - 4\vec k$$